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Hyperbolicity of a 16 Moments Model in the Relativistic Context

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ABSTRACT: *The hyperbolicity requiremet of a recent 16 moments model for RelativisticPolyatomic Gases is here proved for every time direction ξ_α. This is important because asures that the wave propagation ve- locities are finite and not higher than that of light. In addition to important mathematical consequences, this property also shows that the model respects the principle of cause and effect.*

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I. INTRODUCTION

In this article we prove the hyperbolicity requiremet of the field equa- tions proposed in [1] to modelize Polyatomic Gases in the context of Relativistic Extended Thermodynamics. These equations are ex- pressed in terms of the so-called Main Field λ_A (λ , λ_β , $\lambda_{\beta\gamma}$, ν) and they can be written in compact form as

$$\partial_\alpha \left(\frac{\partial h'^\alpha}{\partial \lambda_A} \right) = I^A \quad , \quad \text{or} \quad \frac{\partial^2 h'^\alpha}{\partial \lambda_A \partial \lambda_B} \partial_\alpha \lambda_B = I^A \, . \tag{1}$$

In (1) the 4-potential h'^α is

$$h'^\alpha = - k_B c \int_{\mathbb{R}^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} p^\alpha d\mathcal{I} \, d\vec{P} \, , \tag{2}$$

where

$$f = e^{-1-\frac{\chi}{k_B}} , \quad \text{and} \quad \chi = m\lambda + \left(1 + \frac{\mathcal{I}}{m c^2} \right) \lambda_\beta p^\beta + \tag{3}$$
$$+ \frac{1}{m} \left(1 + \frac{2\mathcal{I}}{m c^2} \right) \lambda_{\alpha_1\alpha_2} p^{\alpha_1} p^{\alpha_2} + \frac{m}{c} \left(1 + \frac{2\mathcal{I}^V}{m c^2} \right) \nu \, ,$$

$\mathcal{I} = \mathcal{I}^R + \mathcal{I}^V$, $d\mathcal{I}$ is an abbreviation for $\phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R d\mathcal{I}^V$. Here $\mathcal{I}^R$ is the contribution to internal energy of a molecule due to the rotational motion and $\mathcal{I}^V$ is the contribution due to the vibrational motion; the functions $\varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)$ measures how much these internal motions influence energy. In any case, eqs. (1) have the symmetric form (but only because we are using the Main Field as independent variables); we will see in sect. 2 that the quadratic form

$$K = \xi_\alpha \frac{\partial^2 h'^\alpha}{\partial \lambda_A \partial \lambda_B} \delta \lambda_A \, \delta \lambda_B \tag{4}$$

is positive definite 8 timelike 4-vector __. Consequently, we have thatthe balance equations (1) aren't only symmetric, but they are also hyperbolic. This property is called "Convexity of Entropy". Thanks to appendix A of [2], this property grants that the characteristic velocities are all real and don' t exceed the speed of light. Even if the article [2] concerned the properties of the article [3], its appendix A was completely general and applicable also to the present balance equations.

II. ONTHECONVEXITYOFENTROPYANDSOME OF ITS CONSEQUENCES

By using the definition (2) of the 4-potential, we have:

$$\frac{\partial h'^{\alpha}}{\partial \lambda_A} = c \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} \frac{\partial \chi}{\partial \lambda_A} p^{\alpha} d\vec{P}\, d\mathcal{I}.$$

Since $\frac{\partial \chi}{\partial \lambda_A}$ does not' t depend on λ_B, it follows

$$\frac{\partial^2 h'^{\alpha}}{\partial \lambda_B \partial \lambda_A} = -\frac{c}{k_B} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} \frac{\partial \chi}{\partial \lambda_B} \frac{\partial \chi}{\partial \lambda_A} p^{\alpha} d\vec{P}\, d\mathcal{I}.$$

By using these results, the quadratic form (4) becomes:

$$K = -\frac{c}{k_B} \xi_{\alpha} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} (\delta \chi)^2 p^{\alpha} d\vec{P} \int_0^{+\infty} d\mathcal{I} \leq 0, \tag{5}$$

$\forall$ time-like unitary 4-vector ξ_{α} and this is true even if ξ_{α} is constant or depending also on x^{α}. In fact, it isn't possible that $\xi_{\alpha} p^{\alpha} = 0$ because they are both time-like 4-vectors (otherwise, in the reference frame where ξ_{α} has the components $\xi_{\alpha} \equiv (1, 0, 0, 0)$ we would have $p^0 = 0$ and, consequently, $p^{\beta} p_{\beta} < 0$ against the fact that $p^{\beta} p_{\beta} = m^2 c^2$). So, for the theorem of existence of zeros for continuos functions, we have that $\xi_{\alpha} p^{\alpha}$ has always the same sign. If $\xi_{\alpha} p^{\alpha} > 0$, then ξ_{α} and p^{α} don't only belong to the same light cone, but they are also both directed towards the future. (Usually in literature, when calculating integrals like those in (2) a change of variables is used with $p^0 = mc \cosh s$, so that $p^0 > 0$ implies that p^{α} has been chosen directed towards the future). Consequently $\xi_{\alpha} p^{\alpha} > 0$ means that we have used the same choice for ξ_{α} and p^{α}.

In any case, even if we choose $\xi_{\alpha} p^{\alpha} < 0$, it suffices to change sign in the definition (2) and we go back to the previous case. It is true that in this way also the balance equations change sign; but it suffices to multiply them by -1 to go back to the previous case.

The condition (5) ensures that the matrix $\frac{\partial^2 h'^{\alpha}}{\partial \lambda_B \partial \lambda_A}$ is negative semi-definite and we prove now that it is negative definite. In fact, we have $K = 0$ if and only if $\delta \chi$ is identically zero, i.e.,

$$\begin{aligned} & m\, \delta \lambda + \left(1 + \frac{\mathcal{I}}{mc^2}\right) p^{\beta} \delta \lambda_{\beta} + \frac{1}{m} \left(1 + \frac{2\mathcal{I}}{mc^2}\right) p^{\beta} p^{\gamma} \delta \lambda_{\beta\gamma} + \\ & + \frac{m}{c} \left(1 + \frac{2\mathcal{I}^V}{mc^2}\right) \delta \nu = 0, \quad \forall p^{\beta}, \mathcal{I}, \mathcal{I}^V, \end{aligned} \tag{6}$$

which are constrained only by $p^{\beta} p_{\beta} = m^2 c^2$. Now, we prove that this condition (6) implies that $\delta \lambda = 0$, $\delta \lambda_{\beta} = 0$, $\delta \lambda_{\beta\gamma} = 0$, $\delta \nu = 0$.

Consequently, we have proved that the matrix $\frac{\partial^2 h'^{\alpha}}{\partial \lambda_B \partial \lambda_A}$ is negative definite, i.e., the convexity of entropy. This result is important because ensures that the differential system is in the symmetric form and the hyperbolicity requirement is automatically satisfied.

PROOF:

The requirement $\forall \mathcal{I}^V$ shows that (6) immediately implies $\delta\nu = 0$. After that, the condition (6) for $P^1 = 0$, $P^2 = 0$, $P^2 = 0$ implies that

$$m\,\delta\,\lambda + \left(1 + \frac{\mathcal{I}}{mc^2}\right) mc\,\delta\,\lambda_0 + \frac{1}{m}\left(1 + \frac{2\mathcal{I}}{mc^2}\right) m^2c^2\,\delta\,\lambda_{00} = 0\,. \quad (7)$$

By subtracting this equation from (6) we obtain

$$\begin{aligned}&\left(1 + \frac{\mathcal{I}}{mc^2}\right)\left[mc(\cosh\, s - 1)\,\delta\,\lambda_0 + mc\sinh\, s\, q^i\,\delta\,\lambda_i\right] + \\ &+ \frac{1}{m}\left(1 + \frac{2\mathcal{I}}{mc^2}\right)\left[m^2c^2\sinh^2\, s\,\delta\,\lambda_{00} + 2m^2c^2\cosh\, s\sinh\, s\, q^i\,\delta\,\lambda_{0i} + \right. \\ &\left. + m^2c^2\sinh^2\, s\, q^i\, q^j\,\delta\,\lambda_{ij}\right] = 0\,, \\ &\forall s \geq 0\,,\, q^i\,:\, q^i q^j\delta_{ij} = -1\,, \\ &\text{where we have chosen}\quad p^0 = mc\cosh\, s\,,\, p^i = mc\, q^i\sinh\, s\,.\end{aligned} \quad (8)$$

We divide now this relation by $\sinh s$ and then take the limit for $s \to 0$ and find

$$\left(1 + \frac{\mathcal{I}}{mc^2}\right) mc\, q^i\,\delta\,\lambda_i + \frac{1}{m}\left(1 + \frac{2\mathcal{I}}{mc^2}\right) 2m^2c^2\, q^i\,\delta\,\lambda_{0i} = 0\,,\ \forall q^i\,,$$

from which

$$\left(1 + \frac{\mathcal{I}}{mc^2}\right)\delta\,\lambda_i = -2\,c\left(1 + \frac{2\mathcal{I}}{mc^2}\right)\delta\,\lambda_{0i}\,. \quad (9)$$

By substituting this result in (8), this becomes

$$\begin{aligned}&\left(1 + \frac{\mathcal{I}}{mc^2}\right) mc(\cosh\, s - 1)\,\delta\,\lambda_0 + \\ &+ \frac{1}{m}\left(1 + \frac{2\mathcal{I}}{mc^2}\right)\left[m^2c^2\sinh^2\, s\,\delta\,\lambda_{00} + 2m^2c^2(\cosh\, s - 1)\sinh\, s\, q^i\,\delta\,\lambda_{0i} + \right. \\ &\left. + m^2c^2\sinh^2\, s\, q^i\, q^j\,\delta\,\lambda_{ij}\right] = 0\,,\end{aligned}$$

whose derivative with respect to s gives

$$\begin{aligned}&\left(1 + \frac{\mathcal{I}}{mc^2}\right)\sinh\, s\,\delta\,\lambda_0 + \\ &+ \left(1 + \frac{2\mathcal{I}}{mc^2}\right)\left\{2c\sinh\, s\cosh\, s\,\delta\,\lambda_{00} + 2c\left[\sinh^2\, s + (\cosh\, s - 1)\cosh\, s\right]\cdot\right. \\ &\left.\cdot\, q^i\,\delta\,\lambda_{0i} + 2c\sinh\, s\cosh\, s\, q^i\, q^j\,\delta\,\lambda_{ij}\right\} = 0\,.\end{aligned} \quad (10)$$

We divide now this relation by $\sinh s$ and then take the limit for $s \to 0$ and find

$$\left(1 + \frac{\mathcal{I}}{mc^2}\right)\delta\,\lambda_0 = -\left(1 + \frac{2\mathcal{I}}{mc^2}\right)\left\{2c\,\delta\,\lambda_{00} + 2c\, q^i\, q^j\,\delta\,\lambda_{ij}\right\}\,, \quad (11)$$

which allows to rewrite (10) as

$$\begin{aligned}&\sinh\, s\,(\cosh\, s - 1)\,\delta\,\lambda_{00} + \left[2\sinh^2\, s + 1 - \cosh\, s\right]\, q^i\,\delta\,\lambda_{0i} + \\ &+ \sinh\, s\,(\cosh\, s - 1)\, q^i\, q^j\,\delta\,\lambda_{ij} = 0\,,\end{aligned} \quad (12)$$

where we have eliminated the common factor $(1+\frac{2\mathcal{I}}{mc^2})$.
We divide now this relation by $\sinh^2 s$ and then take the limit for $s \to 0$ and find

$$\frac{3}{2}\, q^i\, \delta\, \lambda_{0i} = 0 \,\forall\, q^i\,, \quad \text{from which} \quad \delta\, \lambda_{0i} = 0\,. \tag{13}$$

Now, of (12) there remains

$$\delta\, \lambda_{00} + q^i\, q^j\, \delta\, \lambda_{ij} = 0\,, \quad \text{which}$$

for $\vec{q} \equiv (1\,,\,0\,,\,0)$ gives $\delta\,\lambda_{00} + \delta\,\lambda_{11} = 0 \to \delta\,\lambda_{11} = -\,\delta\,\lambda_{00}$,

for $\vec{q} \equiv (0\,,\,1\,,\,0)$ gives $\delta\,\lambda_{00} + \delta\,\lambda_{22} = 0 \to \delta\,\lambda_{22} = -\,\delta\,\lambda_{00}$,

for $\vec{q} \equiv (0\,,\,0\,,\,1)$ gives $\delta\,\lambda_{00} + \delta\,\lambda_{33} = 0 \to \delta\,\lambda_{33} = -\,\delta\,\lambda_{00}$,

for $\vec{q} \equiv (\frac{1}{\sqrt{2}}\,,\,\frac{1}{\sqrt{2}}\,,\,0)$ gives $\delta\,\lambda_{00} + \frac{1}{2}\delta\,\lambda_{11} + \frac{1}{2}\delta\,\lambda_{22} + \delta\,\lambda_{12} = 0$
$\to \delta\,\lambda_{12} = 0$,

for $\vec{q} \equiv (\frac{1}{\sqrt{2}}\,,\,0\,,\,\frac{1}{\sqrt{2}})$ gives $\delta\,\lambda_{00} + \frac{1}{2}\delta\,\lambda_{11} + \frac{1}{2}\delta\,\lambda_{33} + \delta\,\lambda_{13} = 0$
$\to \delta\,\lambda_{13} = 0$,

for $\vec{q} \equiv (0\,,\,\frac{1}{\sqrt{2}}\,,\,\frac{1}{\sqrt{2}})$ gives $\delta\,\lambda_{00} + \frac{1}{2}\delta\,\lambda_{22} + \frac{1}{2}\delta\,\lambda_{33} + \delta\,\lambda_{23} = 0$
$\to \delta\,\lambda_{23} = 0$.

So we have found that

$$\delta\, \lambda_{ij} = -\,\delta\, \lambda_{00}\; \delta_{ij}\,, \tag{14}$$

where δ_{ij} is the Kronecker symbol.
Now we can define

$$\delta\, \lambda_{<\mu\nu>} = \delta\, \lambda_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu}\, g^{\beta\gamma}\, \delta\, \lambda_{\beta\gamma}\,.$$

From (14), (13) we see that the components of $\delta\,\lambda_{<\mu\nu>}$ with $\mu \neq \nu$ are zero, while its components 00, 11, 22, 33 are respectively
$\frac{3}{4}\,\delta\,\lambda_{00} + \frac{1}{4}\,(\delta\,\lambda_{11} + \delta\,\lambda_{22} + \delta\,\lambda_{33})$, $\frac{3}{4}\,\delta\,\lambda_{11} + \frac{1}{4}\,(\delta\,\lambda_{00} - \delta\,\lambda_{22} - \delta\,\lambda_{33})$,
$\frac{3}{4}\,\delta\,\lambda_{22} + \frac{1}{4}\,(\delta\,\lambda_{00} - \delta\,\lambda_{11} - \delta\,\lambda_{33})$, $\frac{3}{4}\,\delta\,\lambda_{33} + \frac{1}{4}\,(\delta\,\lambda_{00} - \delta\,\lambda_{11} - \delta\,\lambda_{22})$,
respectively; these components are zero for (14). In other words, we have $\delta\,\lambda_{<\mu\nu>} = 0$, i.e., $\delta\,\lambda_{\mu\nu} = g_{\mu\nu}\,\delta\, X$ with $\delta\, X = \frac{1}{4}\, g^{\beta\gamma}\,\delta\,\lambda_{\beta\gamma}$.
After that, eq. (11) becomes

$$\left(1+\frac{\mathcal{I}}{mc^2}\right)\,\delta\,\lambda_0 = -\,\left(1+\frac{2\mathcal{I}}{mc^2}\right)\,4c\,\delta\, X\,.$$

Moreover, (9) and (13) give $\delta\,\lambda_i = 0$. Finally, (7) becomes

$$\delta\,\lambda - 3\,\left(1+\frac{2\mathcal{I}}{mc^2}\right)\,c^2\,\delta\, X = 0\,.$$

Now, in the models without rotational and vibrational motion, we have $g^{\beta\gamma}\,\lambda_{\beta\gamma}=0$ which implies $\delta\,X=0$; so the last two relations give $\delta\,\lambda_0=0$, $\delta\,\lambda=0$ and hyperbolicity is proved.
In presence of rotational and vibrational motion, the last two relations must hold $\forall\mathcal{I}$ so that it follows $\delta\,X=0$, $\delta\,\lambda=0$, $\delta\,\lambda_0=0$. Consequently, also for this case hyperbolicity is proved.

III. CONSEQUENCES OF THE ABOVE RESULTS

The inequality (5) is ensured also in the non linear case but, if it holds for every value of the independent variables, then it must hold also when these variables take their equilibrium values. This will allow us to find interesting properties of a set of integrals. In fact, (5) can be written explicitly as

$$K=\xi_\alpha\left[\frac{\partial^2 h'^\alpha}{\partial\lambda^2}\,(\delta\,\lambda)^2+2\,\frac{\partial^2 h'^\alpha}{\partial\lambda\,\partial\lambda_\mu}\,\delta\,\lambda\,\delta\,\lambda_\mu+2\,\frac{\partial^2 h'^\alpha}{\partial\lambda\,\partial\lambda_{\mu\nu}}\,\delta\,\lambda\,\delta\,\lambda_{\mu\nu}+\right.$$
$$+\frac{\partial^2 h'^\alpha}{\partial\lambda_\beta\,\partial\lambda_\mu}\,\delta\,\lambda_\beta\,\delta\,\lambda_\mu+2\,\frac{\partial^2 h'^\alpha}{\partial\lambda_\beta\,\partial\lambda_{\mu\nu}}\,\delta\,\lambda_\beta\,\delta\,\lambda_{\mu\nu}+\frac{\partial^2 h'^\alpha}{\partial\lambda_{\beta\gamma}\,\partial\lambda_{\mu\nu}}\,\delta\,\lambda_{\beta\gamma}\,\delta\,\lambda_{\mu\nu}+$$
$$+\frac{\partial^2 h'^\alpha}{\partial\nu^2}\,(\delta\,\nu)^2+2\,\frac{\partial^2 h'^\alpha}{\partial\nu\,\partial\lambda}\,\delta\,\nu\,\delta\,\lambda+2\,\frac{\partial^2 h'^\alpha}{\partial\nu\,\partial\lambda_\mu}\,\delta\,\nu\,\delta\,\lambda_\mu+$$
$$\left.+2\,\frac{\partial^2 h'^\alpha}{\partial\nu\,\partial\lambda_{\mu\nu}}\,\delta\,\nu\,\delta\,\lambda_{\mu\nu}\right]\,.$$

By calculating the coefficients at equilibrium, it becomes

$$-\,\frac{k_B}{m}\,K_E=\xi_\alpha\left[V_E^\alpha\,(\delta\,\lambda)^2+2\,T_E^{\alpha\mu}\,\delta\,\lambda\,\delta\,\lambda_\mu+2\,A_E^{\alpha\mu\nu}\,\delta\,\lambda\,\delta\,\lambda_{\mu\nu}\right.$$
$$+m\,A_{11}^{\alpha\beta\delta}\,\delta\,\lambda_\beta\,\delta\,\lambda_\delta+2m\,A_{12}^{\alpha\beta\mu\nu}\,\delta\,\lambda_\beta\,\delta\,\lambda_{\mu\nu}+m\,A_{22}^{\alpha\beta\gamma\mu\nu}\,\delta\,\lambda_{\beta\gamma}\,\delta\,\lambda_{\mu\nu}+$$
$$\left.+V_{VV}^\alpha\,(\delta\,\nu)^2+2\,H_{VE}^\alpha\,\delta\,\lambda\,\delta\,\nu+2\,T_V^{\alpha\mu}\,\delta\,\nu\,\delta\,\lambda_\mu+2\,A_V^{\alpha\mu\nu}\,\delta\,\nu\,\delta\,\lambda_{\mu\nu}\right]\,,$$

where the expressions of the tensors in the right hand side are reported in [1]. For the sake of simplicity, we calculate also the coefficients of the differentials in the reference frame where U^α and ξ^α have the components $U^\alpha\equiv(c\,,\,0\,,\,0\,,\,0)$ and $\xi^\alpha(\xi_0\,,\,\xi_1\,,\,0\,,\,0)$ with $\xi_0=\sqrt{1+(\xi_1)^2}$; in any case, we can at the end express again all the results in covariant form replacing ξ_0 and $(\xi_1)^2$ with $\xi_0=\frac{1}{c}\,\xi^\alpha U_\alpha$ and $(\xi_1)^2=\xi_\alpha\xi_\beta h^{\alpha\beta}$.
We define also $W_1=\delta\,\lambda$, $W_2=c\,\delta\,\lambda_0$, $W_3=c^2\,\delta\,\lambda_{00}$, $W_4=\delta\,\lambda_{11}$, $W_5=\delta\,\lambda_{22}+\delta\,\lambda_{33}$, $W_6=\delta\,\nu$, $W_7=\delta\,\lambda_1$, $W_8=c\,\delta\,\lambda_{01}$, $W_9=\delta\,\lambda_{22}-\delta\,\lambda_{33}$, $W_{10}=\delta\,\lambda_2$, $W_{11}=c\,\delta\,\lambda_{02}$, $W_{12}=\delta\,\lambda_{12}$, $W_{13}=\delta\,\lambda_3$, $W_{14}=c\,\delta\,\lambda_{03}$, $W_{15}=\delta\,\lambda_{13}$, $W_{16}=\delta\,\lambda_{23}$ from which it follows

$$\delta\,\lambda_{22}=\frac{1}{2}\,(W_5+W_9)\quad,\quad\delta\,\lambda_{33}=\frac{1}{2}\,(W_5-W_9)\,.$$

In this way our quadratic form becomes

$$-\,\frac{k_B\,c}{m^2}\,K_E=\sum_{a,b=1}^{8}\,\overline{M}^{ab}W_a\,W_b+\sum_{a,b=9}^{11}\,\overline{N}^{ab}W_a\,W_b+\sum_{a,b=12}^{14}\,\overline{N}^{ab}W_a\,W_b+$$
$$+\,\frac{1}{15}\,B_6\,c^2\,\xi_0\left[4\,(W_{16})^2+(Z_9)^2\right]\,,\qquad(15)$$

and, moreover, the matrices $\overline{M}^{ab}$ and $\overline{N}^{ab}$, written in compact form, are

$$\overline{M} = \begin{pmatrix} \xi_0\,\overline{A} & \xi_1\,\overline{B} \\ \xi_1\,\overline{B}^T & \xi_0\,\overline{C} \end{pmatrix},$$

$$\overline{N} = \begin{pmatrix} \frac{1}{3}\,B_4\,c^2\,\xi_0 & \frac{2}{3}\,B_2\,c^2\,\xi_0 & \frac{2}{15}\,B_1\,c\,\xi_1 \\ \frac{2}{3}\,B_2\,c^2\,\xi_0 & \frac{4}{3}\,B_7\,c^2\,\xi_0 & \frac{4}{15}\,B_6\,c\,\xi_1 \\ \frac{2}{15}\,B_1\,c\,\xi_1 & \frac{4}{15}\,B_6\,c\,\xi_1 & \frac{4}{15}\,B_6\,c^2\,\xi_0 \end{pmatrix},$$

where

$$\overline{A} = \begin{pmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0 c^2}{m} & \frac{A_{11}^0 c^2}{m} & \frac{A_{11}^0 c^2}{m} & \frac{H_V c^2}{m} \\ \frac{e}{m} & B_5\,c^2 & B_3\,c^2 & \frac{1}{3}\,B_2 c^2 & \frac{1}{3}\,B_2 c^2 & \frac{B_9}{m} \\ \frac{A_1^0 c^2}{m} & B_3\,c^2 & B_8\,c^2 & \frac{1}{3}\,B_7 c^2 & \frac{1}{3}\,B_7 c^2 & \frac{A_{1V}^0 c^2}{m} \\ \frac{A_{11}^0 c^2}{m} & \frac{1}{3}\,B_2 c^2 & \frac{1}{3}\,B_7 c^2 & \frac{1}{5}\,B_6 c^2 & \frac{1}{15}\,B_6 c^2 & \frac{A_{11V}^0 c^2}{m} \\ \frac{A_{11}^0 c^2}{m} & \frac{1}{3}\,B_2 c^2 & \frac{1}{3}\,B_7 c^2 & \frac{1}{15}\,B_6 c^2 & \frac{2}{15}\,B_6 c^2 & \frac{A_{11V}^0 c^2}{m} \\ \frac{H_V c^2}{m} & \frac{B_9}{m} & \frac{A_{1V}^0 c^2}{m} & \frac{A_{11V}^0 c^2}{m} & \frac{A_{11V}^0 c^2}{m} & \frac{B_{11} c^2}{m} \end{pmatrix},$$

$$\overline{B} = \begin{pmatrix} \frac{pc}{m} & 2\,\frac{A_{11}^0\,c}{m} \\ \frac{1}{3}\,B_4\,c & \frac{2}{3}\,B_2\,c \\ \frac{1}{3}\,B_2\,c & \frac{2}{3}\,B_7\,c \\ \frac{1}{5}\,B_1\,c & \frac{2}{5}\,B_6\,c \\ \frac{1}{15}\,B_1\,c & \frac{2}{15}\,B_6\,c \\ \frac{B_{10}}{m}\,c & 2\,\frac{A_{11V}^0}{m} \end{pmatrix},$$

$$\overline{C} = \begin{pmatrix} \frac{1}{3}\,B_4\,c^2 & \frac{2}{3}\,B_2\,c^2 \\ \frac{2}{3}\,B_2\,c^2 & \frac{4}{3}\,B_7\,c^2 \end{pmatrix}.$$

We note that C is the matrix N in [5] and that the algebraic com-plement of the element in the third line and third coulumn of N is C multiplied by ξ_0.

Now we have proved above in (5) that K is negative definite time- like unitary 4-vector ξ_α and for every value of the variables; so this property holds also if these variables assume their values at equilib- rium. In other words, we have now that K_E is positive defined. This was true in its integral form (5), so it must be true also after calculations of the integrals. So we are sure that the above matrixes

$\tilde{M}$ and $\tilde{N}$ are positive defined (There is no problem for the remaining

parts of K_E because $B_6 > 0$ is evident from its expression in $(A.8)_1$ of [3]) time-like ξ_α. This isn't a new requirement as it may seem apparently from the description in [5]. It is a proved theorem.
But we have now more than this result because we have that the ma- trixes M and N are positive defined also for every value of ξ_1 f= 0.

- Let us begin with M .

For a well know theorem on positive defined matrixes, we have that
$H_p > 0$ for $p = 1, \cdots, 8$ and where H_p denotes the subdeterminant
obtained from $\tilde{M}$ by eliminating its last $8 - p$ lines and its last $8 -$
p coulumns. With the same metodology used in [4] (See also its appendix B which can be applied also to the present case with two additional independent variables), we find that these results $H_p > 0$ for every value of ξ_1 and with $(\xi_0)^2 = 1 + (\xi_1)^2$ can be translated in results expressed in terms only of the elements in the matrixes A , B , C.

• Let us consider now N .

Since ξ_0 C is the principal minor in the last two lines and last two coulumns of M , we have already that it is positive defined. But it is also the principal minor in the first two lines and first two coulumns of N ; so we have only to exploit the property N > 0. To calcu- late it we exchange its third line with the second one; after that we exchange its new second line with the first; after that we exchange its third coulumn with the second; after that we exchange its new second coulumn with the first; after that take out a factor 2 from its new first line and a factor 2 from its new first coulumn. So it becomes

$$|\overline{N}| = \frac{4}{9}\begin{vmatrix} \frac{3}{5}B_6 c^2 \xi_0 & \frac{1}{5}B_1 c\xi_1 & \frac{2}{5}B_6 c\xi_1 \\ \frac{1}{5}B_1 c\xi_1 & \frac{1}{3}B_4 c^2 \xi_0 & \frac{2}{3}B_2 c^2 \xi_0 \\ \frac{2}{5}B_6 c\xi_1 & \frac{2}{3}B_2 c^2 \xi_0 & \frac{4}{3}B_7 c^2 \xi_0 \end{vmatrix} =$$

$$= \frac{4}{9}\begin{vmatrix} \frac{1}{5}B_6 c^2 \xi_0 & \frac{1}{5}B_1 c\xi_1 & \frac{2}{5}B_6 c\xi_1 \\ \frac{1}{5}B_1 c\xi_1 & \frac{1}{3}B_4 c^2 \xi_0 & \frac{2}{3}B_2 c^2 \xi_0 \\ \frac{2}{5}B_6 c\xi_1 & \frac{2}{3}B_2 c^2 \xi_0 & \frac{4}{3}B_7 c^2 \xi_0 \end{vmatrix} + \frac{8}{45}B_6 c^2 (\xi_0)^3 |\overline{C}| > 0,$$

because we have already obtained the property C > 0, while the other determinant is the principal minor of the matrix M where its lines 4,7,8 meet the coulumns 4,7,8. So the property, that N is positive defined, gives no further informations.

IV. An useful change of variables

To compare the present results to those of [4], where less independent variables are present, let us use the following change of variables where A, B, C, $\bar{N}$ are the same matrixes appearing in [4], while

$$E = \begin{pmatrix} -4\,\frac{A^0_{11}\,c^2}{m} & \frac{H_V\,c^2}{m} \\ -\frac{4}{3}\,B_2\,c^2 & \frac{B_9}{m} \\ -\frac{68}{15}\,B_6\,c & \frac{A^0_{1V}\,c^2 + A^0_{11V}}{m} \end{pmatrix},$$

$$F = \begin{pmatrix} -\frac{4}{9}\,B_1\,c & \frac{B_{10}\,c}{m} \\ -\frac{8}{9}\,B_6\,c & \frac{4}{16}\,B_6\,c \\ 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} \frac{16}{9}\,B_6\,c & -4\,\frac{A^0_{11V}\,c^2}{m} \\ -4\,\frac{A^0_{11V}\,c^2}{m} & \frac{B_{11}\,c^2}{m} \end{pmatrix}.$$

Consequently, we have that $-\frac{k_B c}{m^2} K_E$ calculated in $X_7 = 0$, $X_8 = 0$ is exactly that appearing in [4] and also the corresponding matrixes

$$W_1 = X_1\,,\; W_2 = X_2\,,\; W_3 = X_3\,,\; W_4 = X_6 + \frac{1}{3\,c^2}\,X_3 - \frac{4}{3}\,X_7\,,$$
$$W_5 = -\,X_6 + \frac{2}{3\,c^2}\,X_3 - \frac{8}{3}\,X_7\,,\; W_6 = X_8\,,\; W_7 = X_4\,,\; W_8 = X_5\,,$$
$$W_9 = 2\,X_9\,,\; W_{10} = Y_1\,,\; W_{11} = Y_2\,,\; W_{12} = Y_3\,,\; W_{13} = Z_1\,,\; W_{14} = Z_2\,,$$
$$W_{15} = Z_3\,,\; W_{16} = \delta\,\lambda_{23}\,.$$

By using this change of independent variables, our quadratic form (15) transforms into

$$-\,\frac{k_B\,c}{m^2}\,K_E = \sum_{a,b=1}^{8} \bar{M}^{ab} X_a\,X_b + \sum_{a,b=1}^{3} \bar{N}^{ab} Y_a\,Y_b + \sum_{a,b=1}^{3} \bar{N}^{ab} Z_a\,Z_b + \\ + \frac{4}{15}\,B_6\,c^2\,\xi_0\left[(\delta\,\lambda_{23})^2 + (X_7)^2\right],$$

with

$$\bar{M} = \begin{pmatrix} \xi_0\,A & \xi_1\,B & \xi_0\,E \\ \xi_1\,B^T & C & \xi_1\,F \\ \xi_0\,E^T & \xi_1\,F^T & \xi_0\,D \end{pmatrix}, \qquad (16)$$

Consequently, we have that $-\frac{k_B c}{m^2} K_E$ calculated in $X_7 = 0$, $X_8 = 0$ is exactly that appearing in [4] and also the corresponding matrixes are the same. Here we have two further independent variables which contribute to the last line and last coulumn of the matrix (16) (In reality, they are two lines and two coulumns becase E, F and D have two coulumns).

So in the present article we have obtained two further properties, besides those of [4], i.e., that the matrix (16) has positive determinant and also its principal minor obtained by eliminating its last line and its last coulumn has positive determinant.

V. CONCLUSIONS

The present results have a relevant importance to support the gen- uineness of those of the work [1]. In particular they prove that the hyperbolicity requiremet holds, with all its nice consequences like the fact that it implies finite propagation of wave speeds and that they are not higher than that of light. Thus they allow us to have a more than suitable model to describe the trend of relativistic polyatomic gases in presence of internal vibrational and rotational energy. In this way they lay the foundations for building such a model even in the non-relativistic case; this article is in preparation, but some bases for it have already been laid in article [6] which presents its structure and proves that it respects the Galilean relativity principle.

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