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The Galilean Relativity Principle for a 16 Moments Model in Classical E.T. of Polyatomic Gases

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Abstract: *I analize here a 16 moments model for polyatomic gases because this is suggested by the limit, for light speed going to infinity, of the corresponding relativistic model. The Galilean relativity principle is imposed by using a methodology which was introduced some years ago by Pennisi and Ruggeri in the framework of monoatomic gases. It is here proved that it can be applied also for the equations of polyatomic gases. Moreover, some integrals will be calculated which are useful to find the explicit closure of the balance equations.*

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I. Introduction

The present model is suggested by the non relativistic limit of that in the article [1]. A goal realized by this work was to find a relativistic model with 15 moments that draws inspiration from [2], but avoiding to take the traceless part of the third balance law. In this way, one obtains a 15 moments model. Another goal realized in this presentation was to find the relativistic version of another model with 15 moments recently appeared in literature [3] in the classical context. This is different from the previous one in which the internal motion is globally considered, while in [3] the internal motions are separated into the rotational part and the vibrational part. To avoid confusion between these two models, they were compacted in only one, even if at the cost of obtaining a 16 moments model; in any case, the present model can be splitted in those with 15 moments which are its subsystems obtained simply by putting equal to zero the Lagrange multiplier corresponding to one or the other of the new two balance equations. (See [4] for a general treatment of subsystems). In [1] the non-relativistic limit of its balance equations has been found and they are the following (1). To find their closure it is necessary to impose the Galilean Relativity Principle (In the relativistic version this was replaced by the Einstenian Relativity Principle which is easier to impose by simply using the covariant formulation). This is the purpose of the present article thus preparing the ground to find its closure in subsequent works. Obviously, in the subsystem corresponding to [3], the results are not different from those found there; this confirms the present considerations, in addition to the fact that they are included in a larger context, that of the 16 moments. The article [2] has inspired other subsequent works such as [5]-[10]; so also the present one inserts in this framework. In harmony with what has just been said, I analize here the model which has the following balance equations:

$$\begin{aligned}
&\partial_t A_0 + \partial_i A_0^i = 0 \quad , \quad \partial_t A_0^{i_1} + \partial_i A_0^{ii_1} = 0 \quad , \quad \partial_t A_0^{i_1 i_2} + \partial_i A_0^{ii_1 i_2} = P_0^{i_1 i_2} \quad , \\
&\partial_t A_1 + \partial_i A_1^i = 0 \quad , \quad \partial_t A_1^{i_1} + \partial_i A_1^{ii_1} = P_1^{i_1} \quad , \\
&\partial_t A_2 + \partial_i A_2^i = P_2 \quad , \quad \partial_t H_V^{ll} + \partial_i H_V^{ill} = P_V^{ll} \, .
\end{aligned} \tag{1}$$

Here A_0, A_0^i, A_1, A_1^i are the densities of mass, momentum and energy, while $(1)_{1,2,4}$ are their conservation laws of mass.
Moreover, we have

$$\begin{aligned}
A_0 &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi}, \\
A_0^{i_1} &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \xi^{i_1} \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi}, \\
A_0^{i_1 i_2} &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \xi^{i_1} \, \xi^{i_2} \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi}, \\
A_1 &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \left(\xi^2 + \frac{2\mathcal{I}}{m} \right) \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi}, \\
A_1^{i_1} &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \left(\xi^2 + \frac{2\mathcal{I}}{m} \right) \xi^{i_1} \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi}, \\
A_2 &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \left(\xi^4 + \frac{4\mathcal{I}}{m} \xi^2 \right) \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi}, \\
H_V^{ll} &= 2 \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \mathcal{I}^V \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi}.
\end{aligned} \tag{2}$$

Here $\mathcal{I}^R$ and $\mathcal{I}^V$ are the internal energies due to rotation and vibration respectively, $\varphi(\mathcal{I}^R)$ and $\psi(\mathcal{I}^V)$ are their measures depending on the particular gas under consideration, while $f(x^i, t, \xi^j, \mathcal{I}^R, \mathcal{I}^V)$ is the distribution function.
The above quantities are the independent variables. Some of the fluxes are equal to the independent variable in the subsequent equation; the remaining ones are

$$\begin{aligned}
A_0^{ii_1i_2} &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, \xi^i\, \xi^{i_1}\, \xi^{i_2}\, \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi}, \\
A_1^{ii_1} &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, \left(\xi^2 + \frac{2\mathcal{I}}{m}\right) \xi^i\, \xi^{i_1}\, \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi}, \\
A_2^{i} &= m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, \left(\xi^4 + \frac{4\mathcal{I}}{m}\xi^2\right) \xi^i\, \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi}, \\
H_V^{ill} &= 2 \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, \xi^i\, \mathcal{I}^V\, \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi}.
\end{aligned} \tag{3}$$

The entropy principle for the balance equations (1) amounts in assuming the existence of the 4-potentials h', h'^i and of the Lagrange multipliers λ, λ_{i_1}, $\lambda_{i_1i_2}$, μ, μ_{i_1}, ν such that

$$A_0 = \frac{\partial h'}{\partial \lambda}\,,\; A_0^{i_1} = \frac{\partial h'}{\partial \lambda_{i_1}}\,,\; A_0^{i_1i_2} = \frac{\partial h'}{\partial \lambda_{i_1i_2}}\,,\; A_1 = \frac{\partial h'}{\partial \mu}\,,\; A_1^{i_1} = \frac{\partial h'}{\partial \mu_{i_1}}\,,\; A_2 = \frac{\partial h'}{\partial \nu}\,,\; H_V^{ll} = \frac{\partial h'}{\partial \mu_V}\,,$$

$$A_0^{ii_1i_2} = \frac{\partial h'^i}{\partial \lambda_{i_1i_2}}\,,\; A_1^{ii_1} = \frac{\partial h'^i}{\partial \mu_{i_1}}\,,\; A_2^{i} = \frac{\partial h'^i}{\partial \nu}\,,\; H_V^{ill} = \frac{\partial h'^i}{\partial \mu_V}\,, \tag{4}$$

$$\frac{\partial h'}{\partial \lambda_i} = \frac{\partial h'^i}{\partial \lambda}\,,\; \frac{\partial h'}{\partial \lambda_{ij}} = \frac{\partial h'^i}{\partial \lambda_j}\,,\; \frac{\partial h'}{\partial \mu_i} = \frac{\partial h'^i}{\partial \mu}\,,\; \frac{\partial h'^{[i}}{\partial \lambda_{j]}} = 0\,,\; \frac{\partial h'^{[i}}{\partial \lambda_{i_1]i_2}} = 0\,,\; \frac{\partial h'^{[i}}{\partial \mu_{i_1]}} = 0\,. \tag{5}$$

Here the eqs. $(5)_{1-3}$ express the fact that some of the fluxes are equal to the independent variable in the subsequent equation, while eq. $(5)_4$ expresses the fact that the left hand side of eq. $(5)_2$ is symmetric, so also the right hand side must be symmetric. Finally, $(5)_{5,6}$ express the fact that $A_0^{ii_1i_2}$ and $A_1^{ii_1}$, in their kinetic formulation (3), are symmetric tensors.

II. The Galilean Relativity Principle

We impose now the the Galilean relativity principle using the methodology introduced in [11] in the framework of monoatomic gases. We prove here that it can be applied also for the equations of polyatomic gases.
Firstly, we want to see how our variables change with a change of a reference frame moving with respect to the other with a translational rectilinear uniform motion whose velocity is v_τ^i; so we indicate with a supplementary index "a" a quantity referred to the absolute reference frame and with a supplementary index "r" a quantity referred to the relative reference frame.
We obtain

$$\begin{aligned}
&A_{0a} = A_{0r}\,,\; A_{0a}^{i_1} = A_{0r}^{i_1} + A_{0r} v_\tau^{i_1}\,,\; A_{0a}^{i_1i_2} = A_{0r}^{i_1i_2} + 2A_{0r}^{(i_1} v_\tau^{i_2)} + A_{0r} v_\tau^{i_1} v_\tau^{i_2}\,, \\
&A_{1a} = A_{1r} + 2\,A_{0r}^{h} v_{h\tau} + A_{0r} v_\tau^2\,, \\
&A_{1a}^{i_1} = A_{1r}^{i_1} + A_{1r} v_\tau^{i_1} + 2A_{0r}^{hi_1} v_{h\tau} + 2A_{0r}^{h} v_{h\tau} v_\tau^{i_1} + A_{0r}^{i_1} v_\tau^2 + A_{0r} v_\tau^2 v_\tau^{i_1}\,, \\
&A_{2a} = A_{2r} + 4A_{1r}^{h} v_{h\tau} + 2A_{1r} v_\tau^2 + 4A_{0r}^{hk} v_{h\tau} v_{k\tau} + 4A_{0r}^{h} v_{h\tau} v_\tau^2 + A_{0r} v_\tau^4\,, \\
&H_{Va}^{ll} = H_{Vr}^{ll}\,, \\
&A_{0a}^{ii_1i_2} - v_\tau^i A_{0a}^{i_1i_2} = A_{0r}^{ii_1i_2} + 2A_{0r}^{i(i_1} v_\tau^{i_2)} + A_{0r}^{i} v_\tau^{i_1} v_\tau^{i_2}\,, \\
&A_{1a}^{ii_1} - v_\tau^i A_{1a}^{i_1} = A_{1r}^{ii_1} + A_{1r}^{i} v_\tau^{i_1} + 2\,A_{0r}^{ihi_1} v_{h\tau} + 2A_{0r}^{ih} v_{h\tau} v_\tau^{i_1} + A_{0r}^{ii_1} v_\tau^2 + A_{0r}^{i} v_\tau^{i_1} v_\tau^2\,, \\
&A_{2a}^{i} - v_\tau^i A_{2a} = A_{2r}^{i} + 4A_{1r}^{ih} v_{h\tau} + 2A_{1r}^{i} v_\tau^2 + 4A_{0r}^{ihk} v_{h\tau} v_{k\tau} + 4A_{0r}^{ih} v_{h\tau} v_\tau^2 + A_{0r}^{i} v_\tau^4\,, \\
&H_{Va}^{ill} - v_\tau^i H_{Va}^{ll} = H_{Vr}^{ill}\,.
\end{aligned} \tag{6}$$

To prove these relations it suffices to use eqs. (2) and (3). By substituting in them ξ^i with $\xi^i_a = \xi^i_r + v^i_\tau$ we find A_{0a}, $A^{i_1}_{0a}$, $A^{i_1 i_2}_{0a}$, A_{1a}, $A^{i_1}_{1a}$, A_{2a}, $A^{ii_1i_2}_{0a}$, $A^{ii_1}_{1a}$, A^{i}_{2a} respectively (we recall that $d\vec{\xi}_a = d\vec{\xi}_r$ because the Jacobian of this change of integration variables is equal to 1); by substituting in them ξ^i with ξ^i_r we find A_{0r}, $A^{i_1}_{0r}$, $A^{i_1i_2}_{0r}$, A_{1r}, $A^{i_1}_{1r}$, A_{2r}, $A^{ii_1i_2}_{0r}$, $A^{ii_1}_{1r}$, A^{i}_{2r} respectively. By substituting these expressions in (6) we see that they are identically satisfied $\forall\, v^i_\tau$ and this completes their proof.

Now we want to obtain the transformation law of the Lagrange multipliers. From $(4)_{1-6}$, we have $d\,h'_a =$

$$= A_{0a}\, d\,\lambda_a + A^{i_1}_{0a}\, d\,\lambda_{i_1a} + A^{i_1i_2}_{0a}\, d\,\lambda_{i_1i_2a} + A_{1a}\, d\,\mu_a + A^{i_1}_{1a}\, d\,\mu_{i_1a} + A_{2a}\, d\,\nu_a + H^{ll}_{V\,a}\, d\,\mu_{V\,a} =$$

$$= A_{0r}\, d\,\lambda_a + \left(A^{i_1}_{0r} + A_{0r}v^{i_1}_\tau\right) d\,\lambda_{i_1a} + \left(A^{i_1i_2}_{0r} + 2A^{(i_1}_{0r} v^{i_2)}_\tau + A_{0r}v^{i_1}_\tau v^{i_2}_\tau\right) d\,\lambda_{i_1i_2a} +$$

$$+ \left(A_{1r} + 2\,A^{h}_{0r} v_{h\tau} + A_{0r}v^2_\tau\right) d\,\mu_a +$$

$$+ \left(A^{i_1}_{1r} + A_{1r}v^{i_1}_\tau + 2A^{hi_1}_{0r} v_{h\tau} + 2A^{h}_{0r} v_{h\tau}v^{i_1}_\tau + A^{i_1}_{0r}v^2_\tau + A_{0r}v^2_\tau v^{i_1}_\tau\right) d\,\mu_{i_1a} +$$

$$+ \left(A_{2r} + 4A^{h}_{1r} v_{h\tau} + 2A_{1r}v^2_\tau + 4A^{hk}_{0r} v_{h\tau}v_{k\tau} + 4A^{h}_{0r} v_{h\tau}v^2_\tau + A_{0r}v^4_\tau\right) d\,\mu_a + A^{ll}_{Vr}\, d\,\mu_{V\,a} =$$

$$= A_{0r}\, d\, \left(\lambda_a + v^{i_1}_\tau \lambda_{i_1a} + v^{i_1}_\tau v^{i_2}_\tau \lambda_{i_1i_2a} + v^2_\tau\, \mu_a + v^2_\tau v^{i_1}_\tau\, \mu_{i_1a} + v^4_\tau \nu_a\right) +$$

$$+ A^{i_1}_{0r}\, d\, \left(\lambda_{i_1a} + 2v^{i_2}_\tau \lambda_{i_1i_2a} + 2\, v_{i_1\tau}\mu_a + 2v_{i_1\tau}v^{i_2}_\tau \mu_{i_2a} + v^2_\tau \mu_{i_1a} + 4v^2_\tau v_{i_1\tau}\nu_a\right) +$$

$$+ A^{i_1i_2}_{0r}\, d\, \left(\lambda_{i_1i_2a} + 2v_{i_1\tau}\mu_{i_2a} + 4v_{i_1\tau}v_{i_2\tau}\nu_a\right) + A_{1r}\, d\, \left(\mu_a + v^{i_1}_\tau \mu_{i_1a} + 2v^2_\tau \nu_a\right) +$$

$$+ A^{i_1}_{1r}\, d\, \left(\mu_{i_1a} + 4v_{i_1\tau}\nu_a\right) + A_{2r}\, d\,\nu_a + A^{ll}_{Vr}\, d\,\mu_{V\,a}\,,$$

where, in the passage after the second line, we have used eqs. (6).

Since h' is a scalar function, this quantity must be equal to

$$d\,h'_r = A_{0r}\, d\,\lambda_r + A^{i_1}_{0r}\, d\,\lambda_{i_1r} + A^{i_1i_2}_{0r}\, d\,\lambda_{i_1i_2r} + A_{1r}\, d\,\mu_r + A^{i_1}_{1r}\, d\,\mu_{i_1r} + A_{2r}\, d\,\nu_r + A^{ll}_{Vr}\, d\,\mu_{V\,r}\,.$$

By comparing the two expresions, we deduce that

$$\begin{aligned}
&\lambda_r = \lambda_a + v^{i_1}_\tau \lambda_{i_1a} + v^{i_1}_\tau v^{i_2}_\tau \lambda_{i_1i_2a} + v^2_\tau\, \mu_a + v^2_\tau v^{i_1}_\tau\, \mu_{i_1a} + v^4_\tau \nu_a\,,\\
&\lambda_{i_1r} = \lambda_{i_1a} + 2v^{i_2}_\tau \lambda_{i_1i_2a} + 2\, v_{i_1\tau}\mu_a + 2v_{i_1\tau}v^{i_2}_\tau \mu_{i_2a} + v^2_\tau \mu_{i_1a} + 4v^2_\tau v_{i_1\tau}\nu_a\,,\\
&\lambda_{i_1i_2r} = \lambda_{i_1i_2a} + 2v_{\tau(i_1}\mu_{i_2)a} + 4v_{i_1\tau}v_{i_2\tau}\nu_a\,,\\
&\mu_r = \mu_a + v^{i_1}_\tau \mu_{i_1a} + 2v^2_\tau \nu_a\,,\\
&\mu_{i_1r} = \mu_{i_1a} + 4v_{i_1\tau}\nu_a\,,\\
&\nu_r = \nu_a\\
&\mu_{V\,r} = \mu_{V\,a}\,.
\end{aligned} \tag{7}$$

This is the requested transformation of the Lagrange multipliers.

This system can be inverted and gives

$$\begin{aligned}
&\lambda_a = \lambda_r - v^{i_1}_\tau \lambda_{i_1r} + v^{i_1}_\tau v^{i_2}_\tau \lambda_{i_1i_2r} + v^2_\tau\, \mu_r - v^2_\tau v^{i_1}_\tau\, \mu_{i_1r} + v^4_\tau \nu_r\,,\\
&\lambda_{i_1a} = \lambda_{i_1r} - 2v^{i_2}_\tau \lambda_{i_1i_2r} - 2\, v_{i_1\tau}\mu_r + 2v_{i_1\tau}v^{i_2}_\tau \mu_{i_2r} + v^2_\tau \mu_{i_1r} - 4v^2_\tau v_{i_1\tau}\nu_r\,,\\
&\lambda_{i_1i_2a} = \lambda_{i_1i_2r} - 2v_{\tau(i_1}\mu_{i_2)r} + 4v_{i_1\tau}v_{i_2\tau}\nu_r\,,\\
&\mu_a = \mu_r - v^{i_1}_\tau \mu_{i_1r} + 2v^2_\tau \nu_r\,,\\
&\mu_{i_1a} = \mu_{i_1r} - 4\, v_{i_1\tau}\, \nu_r\,,\\
&\nu_a = \nu_r\\
&\mu_{V\,a} = \mu_{V\,r}\,,
\end{aligned} \tag{8}$$

that is, again the system (7), but with the suffixes "a" and "r" exchanged and with $-\,v_{i\tau}$ instead of $v_{i\tau}$. Now, for the sequel, it is important to prove that the quantity

$$\chi = \lambda + \lambda_{i_1}\xi^{i_1} + \lambda_{i_1i_2}\xi^{i_1}\xi^{i_2} + \mu\left(\xi^2 + \frac{2\mathcal{I}}{m}\right) + \mu_{i_1}\xi^{i_1}\left(\xi^2 + \frac{2\mathcal{I}}{m}\right) + \nu\left(\xi^4 + \frac{4\mathcal{I}}{m}\xi^2\right) + 2\mu_V\frac{\mathcal{I}^V}{m}\,, \tag{9}$$

is a scalar, that is, $\chi_a = \chi_r$.
This is a consequence of (7) because

$$\chi_r = \lambda_r + \lambda_{i_1 r}\left(\xi^{i_1} - v_{\mathcal{T}}^{i_1}\right) + \lambda_{i_1 i_2 r}\left(\xi^{i_1} - v_{\mathcal{T}}^{i_1}\right)\left(\xi^{i_2} - v_{\mathcal{T}}^{i_2}\right) + \mu_r\left(\xi^h \xi_h - 2\xi^h v_{h\mathcal{T}} + v_{\mathcal{T}}^2 + \frac{2\mathcal{I}}{m}\right) +$$
$$+ \mu_{i_1 r}\left(\xi^{i_1} - v_{\mathcal{T}}^{i_1}\right)\left(\xi^h \xi_h - 2\xi^h v_{h\mathcal{T}} + v_{\mathcal{T}}^2 + \frac{2\mathcal{I}}{m}\right) + \nu_r\left[\left(\xi^h \xi_h\right)^2 + 4\left(\xi^h v_{h\mathcal{T}}\right)^2 + v_{\mathcal{T}}^4 -\right.$$
$$\left. 4\left(\xi^h \xi_h\right)\left(\xi^k v_{k\mathcal{T}}\right) + 2\left(\xi^h \xi_h\right) v_{\mathcal{T}}^2 - 4\left(\xi^h v_{h\mathcal{T}}\right) v_{\mathcal{T}}^2 + \frac{4\mathcal{I}}{m}\left(\xi^h \xi_h - 2\xi^h v_{h\mathcal{T}} + v_{\mathcal{T}}^2\right)\right] + 2\mu_{V r}\frac{\mathcal{I}^V}{m},$$

where we have indicated the absolute velocity with ξ^i. By substituting here λ_r, $\lambda_{i_1 r}$, $\lambda_{i_1 i_2 r}$, μ_r, $\mu_{i_1 r}$, ν_r from eqs. (7), we see that this quantity is identically equal to

$$\chi_a = \lambda_a + \lambda_{i_1 a}\xi^{i_1} + \lambda_{i_1 i_2 a}\xi^{i_1}\xi^{i_2} + \mu_a\left(\xi^2 + \frac{2\mathcal{I}}{m}\right) + \mu_{i_1 a}\xi^{i_1}\left(\xi^2 + \frac{2\mathcal{I}}{m}\right) +$$
$$+\nu_a\left(\xi^4 + \frac{4\mathcal{I}}{m}\xi^2\right) + 2\mu_{V a}\frac{\mathcal{I}^V}{m},$$

$\forall\, v_{\mathcal{T}}^i$. This fact confirms that χ is a scalar.
Another useful consequence of (7) is

$$\begin{aligned} &\frac{\partial \lambda_r}{\partial v_{\mathcal{T}}^i} = \lambda_{ir}, \ \frac{\partial \lambda_{i_1 r}}{\partial v_{\mathcal{T}}^i} = 2\lambda_{i_1 i r} + 2\delta_{i i_1}\mu_r, \ \frac{\partial \lambda_{i_1 i_2 r}}{\partial v_{\mathcal{T}}^i} = 2\delta_{i(i_1}\mu_{i_2)r}, \\ &\frac{\partial \mu_r}{\partial v_{\mathcal{T}}^i} = \mu_{ir}, \ \frac{\partial \mu_{i_1 r}}{\partial v_{\mathcal{T}}^i} = 4\delta_{i i_1}\nu_r, \ \frac{\partial \nu_r}{\partial v_{\mathcal{T}}^i} = 0, \ \frac{\partial \mu_{V r}}{\partial v_{\mathcal{T}}^i} = 0. \end{aligned} \tag{10}$$

Similarly, a consequence of (8) is

$$\begin{aligned} &\frac{\partial \lambda_a}{\partial v_{\mathcal{T}}^i} = -\lambda_{ia}, \ \frac{\partial \lambda_{i_1 a}}{\partial v_{\mathcal{T}}^i} = -2\lambda_{i_1 i a} - 2\delta_{i i_1}\mu_a, \ \frac{\partial \lambda_{i_1 i_2 a}}{\partial v_{\mathcal{T}}^i} = -2\delta_{i(i_1}\mu_{i_2)a}, \\ &\frac{\partial \mu_a}{\partial v_{\mathcal{T}}^i} = -\mu_{ia}, \ \frac{\partial \mu_{i_1 a}}{\partial v_{\mathcal{T}}^i} = -4\delta_{i i_1}\nu_a, \ \frac{\partial \nu_a}{\partial v_{\mathcal{T}}^i} = 0, \ \frac{\partial \mu_{V a}}{\partial v_{\mathcal{T}}^i} = 0. \end{aligned} \tag{11}$$

Up to now, we have not imposed the Gailean relativity principle. One of the constraints of this principle is the following one: Let us think of h' as a function of λ_r, $\lambda_{i_1 r}$, $\lambda_{i_1 i_2 r}$, μ_r, $\mu_{i_1 r}$, ν_r, μ_{Vr}; if we substitute here (7), it becomes a function of λ_a, $\lambda_{i_1 a}$, $\lambda_{i_1 i_2 a}$, μ_a, $\mu_{i_1 a}$, ν_a, μ_{Va} and of $v_{\mathcal{T}}^i$. Well, the principle states that this new function must not depend on $v_{\mathcal{T}}^i$. In other words, we must have $\frac{\partial h'}{\partial v_{\mathcal{T}}^i} = 0$, because

$$h = \int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} -k_B f \ln f \, \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi} \quad , \quad h' = -h + \lambda_A F^A,$$

(where k_B is the Boltzmann constant) and $\lambda_A F^A$ is independent on the reference frame, as it can be seen from $(6)_{1-6}$ and (8). By using (11), this condition becomes

$$\frac{\partial h'}{\partial \lambda}\lambda_i + 2\frac{\partial h'}{\partial \lambda_{i_1}}\left(\lambda_{i_1 i} + \delta_{i_1 i}\mu\right) + 2\frac{\partial h'}{\partial \lambda_{i i_2}}\mu_{i_2} + \frac{\partial h'}{\partial \mu}\mu_i + 4\frac{\partial h'}{\partial \mu_i}\nu = 0, \tag{12}$$

where we have omitted the suffix a because we aim to find conditions in the absolute reference frame.

Another constraint of the Galilean relativity principle is that $h'^i_a - h' v^i_\tau$ must not depend on v^j_τ, i.e., $\frac{\partial (h'^i_a - h' v^i_\tau)}{\partial v^j_\tau} = 0$, because

$$h^i = \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} -k_B\, f \,\ln\, f\, \xi^i\, \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi} \quad , \quad h'^i = -\,h^i + \lambda_A\, F^{iA}\,,$$

from which it follows

$$h^i_a = \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} -k_B\, f \,\ln\, f\, \left[(\xi^i - v^i_\tau) + v^i_\tau \right] \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi} = h^i_r + h\, v^i_\tau\,,$$

$$h'^i_a - h' v^i_\tau = -\,h^i_r + \lambda_A \left(F^{iA} - F^A v^i_\tau \right)\,,$$

and $\lambda_A \left(F^{iA} - F^A v^i_\tau \right)$ is independent on the reference frame, as it can be seen from (6) and (8). . By using (11), this other condition becomes

$$\frac{\partial\, h'^i_a}{\partial\, \lambda_a}\, \lambda_{ja} + 2\, \frac{\partial\, h'^i_a}{\partial\, \lambda_{ha}}\, (\lambda_{hja} + \delta_{hj}\, \mu_a) + 2\, \frac{\partial\, h'^i_a}{\partial\, \lambda_{jha}}\, \mu_{ha} + \frac{\partial\, h'^i_a}{\partial\, \mu_a}\, \mu_{ja} + 4\, \frac{\partial\, h'^i_a}{\partial\, \mu_{ja}}\, \nu_a + h'\, \delta_{ij} = 0\,. \tag{13}$$

We can now see that for every h' and h'^i satisfying (5), (12), (13), all the other conditions (6) are satisfied, while the conditions (5) are independent from the reference frame.

- Let us begin with eqs. $(6)_{1-7}$. We have

$$A_{0a} = \frac{\partial\, h'}{\partial\, \lambda_a} = \frac{\partial\, h'}{\partial\, \lambda_{Ar}}\, \frac{\partial\, \lambda_{Ar}}{\partial\, \lambda_a} \overset{*}{=} \frac{\partial\, h'}{\partial\, \lambda_r} = A_{0r}\,,$$

where in the passage denoted with $\overset{*}{=}$ we have used (7). Similarly,

where in the passage denoted with $\overset{*}{=}$ we have used (7). Similarly,

$$A^{i_1}_{0a} = \frac{\partial\, h'}{\partial\, \lambda_{i_1 a}} = \frac{\partial\, h'}{\partial\, \lambda_{Ar}}\, \frac{\partial\, \lambda_{Ar}}{\partial\, \lambda_{i_1 a}} \overset{*}{=} \frac{\partial\, h'}{\partial\, \lambda_r}\, v^{i_1}_\tau + \frac{\partial\, h'}{\partial\, \lambda_{i_1 r}} = A_0\, v^{i_1}_\tau + A^{i_1}_{0r}\,.$$

$$A^{i_1 i_2}_{0a} = \frac{\partial\, h'}{\partial\, \lambda_{i_1 i_2 a}} = \frac{\partial\, h'}{\partial\, \lambda_{Ar}}\, \frac{\partial\, \lambda_{Ar}}{\partial\, \lambda_{i_1 i_2 a}} \overset{*}{=} \frac{\partial\, h'}{\partial\, \lambda_r}\, v^{i_1}_\tau v^{i_2}_\tau + 2 \frac{\partial\, h'}{\partial\, \lambda_{j_1 r}}\, v^{(i_1}_\tau \delta^{i_2)}_{j_1} + \frac{\partial\, h'}{\partial\, \lambda_{i_1 i_2 r}} =$$
$$= A_0\, v^{i_1}_\tau v^{i_2}_\tau + 2\; A^{(i_1}_{0r} v^{i_2)}_\tau + A^{i_1 i_2}_{0r}\,.$$

$$A_{1a} = \frac{\partial\, h'}{\partial\, \mu_a} = \frac{\partial\, h'}{\partial\, \lambda_{Ar}}\, \frac{\partial\, \lambda_{Ar}}{\partial\, \mu_a} \overset{*}{=} \frac{\partial\, h'}{\partial\, \lambda_r}\, v^2_\tau + 2 \frac{\partial\, h'}{\partial\, \lambda_{j_1 r}}\, v_{j_1 \tau} + \frac{\partial\, h'}{\partial\, \mu_r} =$$
$$= A_0\; v^2_\tau + 2\, A^{j_1}_{0r} v_{j_1 \tau} + A_{1r}\,.$$

$$A^{i_1}_{1a} = \frac{\partial\, h'}{\partial\, \mu_{i_1 a}} = \frac{\partial\, h'}{\partial\, \lambda_{Ar}}\, \frac{\partial\, \lambda_{Ar}}{\partial\, \mu_{i_1 a}} \overset{*}{=} \frac{\partial\, h'}{\partial\, \lambda_r}\, v^2_\tau v^{i_1}_\tau + \frac{\partial\, h'}{\partial\, \lambda_{j_1 r}} \left(2\, v_{j_1 \tau} v^{i_1}_\tau + v^2_\tau \delta^{i_1}_{j_1} \right) + \frac{\partial\, h'}{\partial\, \lambda_{j_1 j_2 r}}\, 2\, v_{\tau (j_1} \delta^{i_1}_{j_2)} +$$
$$+ \frac{\partial\, h'}{\partial\, \mu_r} v^{i_1}_\tau + \frac{\partial\, h'}{\partial\, \mu_{i_1 r}} = A_0\, v^{i_1}_\tau\, v^2_\tau + A^{j_1}_{0r} \left(2\, v_{j_1 \tau} v^{i_1}_\tau + v^2_\tau \delta^{i_1}_{j_1} \right) + A^{j_1 j_2}_{0r}\, 2\, v_{\tau (j_1} \delta^{i_1}_{j_2)} + A_{1r} v^{i_1}_\tau + A^{i_1}_{1r}\,.$$

$$A_{2a} = \frac{\partial\, h'}{\partial\, \nu_a} = \frac{\partial\, h'}{\partial\, \lambda_{Ar}}\, \frac{\partial\, \lambda_{Ar}}{\partial\, \nu_a} \overset{*}{=} \frac{\partial\, h'}{\partial\, \lambda_r}\, v^4_\tau + \frac{\partial\, h'}{\partial\, \lambda_{j_1 r}}\, 4\, v_{j_1 \tau} v^2_\tau + \frac{\partial\, h'}{\partial\, \lambda_{j_1 j_2 r}}\, 4\, v_{\tau j_1} v_{\tau j_2} + \frac{\partial\, h'}{\partial\, \mu_r}\, 2\, v^2_\tau +$$
$$+ \frac{\partial\, h'}{\partial\, \mu_{j_1 r}}\, 4\, v_{\tau j_1} + \frac{\partial\, h'}{\partial\, \nu_r} = A_0\, v^4_\tau + A^{j_1}_{0r}\, 4\, v_{j_1 \tau} v^2_\tau + A^{j_1 j_2}_{0r}\, 4\, v_{\tau j_1} v_{\tau j_2} + A_{1r}\, 2\, v^2_\tau + A^{j_1}_{1r}\, 4\, v_{\tau j_1} + A_{2r}\,,$$

$$H^{ll}_{Va} = \frac{\partial\, h'}{\partial\, \mu_{V a}} = \frac{\partial\, h'}{\partial\, \lambda_{Ar}}\, \frac{\partial\, \lambda_{Ar}}{\partial\, \mu_{V a}} \overset{*}{=} \frac{\partial\, h'}{\partial\, \mu_{V r}} = H^{ll}_{Vr}\,.$$

- Let us now prove that $(5)_1$ is independent from the reference frame. We have

$$\frac{\partial}{\partial v_\tau^j}\frac{\partial h'}{\partial \lambda_{ia}} = \frac{\partial^2 h'}{\partial \lambda_{Aa}\,\partial \lambda_{ia}}\frac{\partial \lambda_{Aa}}{\partial v_\tau^j} \stackrel{*}{=} -\frac{\partial^2 h'}{\partial \lambda_a\,\partial \lambda_{ia}}\,\lambda_{ja} - 2\frac{\partial^2 h'}{\partial \lambda_{j_1 a}\,\partial \lambda_{ia}}\left(\lambda_{j_1 ja} + \delta_{j_1 j}\,\mu_a\right) - \frac{\partial^2 h'}{\partial \lambda_{jj_2 a}\,\partial \lambda_{ia}}\,2\,\mu_{j_2 a} - \frac{\partial^2 h'}{\partial \mu_a\,\partial \lambda_{ia}}\,\mu_{ja} - 4\frac{\partial^2 h'}{\partial \mu_{ja}\,\partial \lambda_{ia}}\,\nu_a \stackrel{**}{=} \frac{\partial h'}{\partial \lambda_a}\,\delta^{ij}\,,$$

where in the passage denoted with $\stackrel{*}{=}$ we have used (11) and in that denoted with $\stackrel{**}{=}$ we have used the derivative of eq. (12) with respect to λ_{ia}. Similarly, we have

$$\frac{\partial}{\partial v_\tau^j}\frac{\partial h_a'^i}{\partial \lambda_a} = \frac{\partial^2 h_a'^i}{\partial \lambda_{Aa}\,\partial \lambda_a}\frac{\partial \lambda_{Aa}}{\partial v_\tau^j} \stackrel{*}{=} -\frac{\partial^2 h_a'^i}{\partial \lambda_a^2}\,\lambda_{ja} - 2\frac{\partial^2 h_a'^i}{\partial \lambda_{j_1 a}\,\partial \lambda_a}\left(\lambda_{j_1 ja} + \delta_{j_1 j}\,\mu_a\right) - \frac{\partial^2 h_a'^i}{\partial \lambda_{jj_2 a}\,\partial \lambda_a}\,2\,\mu_{j_2 a} - \frac{\partial^2 h_a'^i}{\partial \mu_a\,\partial \lambda_a}\,\mu_{ja} - 4\frac{\partial^2 h_a'^i}{\partial \mu_{ja}\,\partial \lambda_a}\,\nu_a \stackrel{**}{=} \frac{\partial h'}{\partial \lambda_a}\,\delta^{ij}\,,$$

where in the passage denoted with $\stackrel{*}{=}$ we have used (11) and in that denoted with $\stackrel{**}{=}$ we have used the derivative of eq. (13) with respect to λ_a. Consequently, we have

$$\frac{\partial}{\partial v_\tau^j}\left(\frac{\partial h'}{\partial \lambda_{ia}} - \frac{\partial h_a'^i}{\partial \lambda_a}\right) = 0\,.$$

This proves that $(5)_1$ doesn't depend on v_τ^j, i.e., doesn't depend on the reference frame.

- Let us now prove that $(5)_2$ is independent from the reference frame. We have

$$\frac{\partial}{\partial v_\tau^j}\frac{\partial h'}{\partial \lambda_{ii_1 a}} = \frac{\partial^2 h'}{\partial \lambda_{Aa}\,\partial \lambda_{ii_1 a}}\frac{\partial \lambda_{Aa}}{\partial v_\tau^j} \stackrel{*}{=} -\frac{\partial^2 h'}{\partial \lambda_a\,\partial \lambda_{ii_1 a}}\,\lambda_{ja} - 2\frac{\partial^2 h'}{\partial \lambda_{j_1 a}\,\partial \lambda_{ii_1 a}}\left(\lambda_{j_1 ja} + \delta_{j_1 j}\,\mu_a\right) - \frac{\partial^2 h'}{\partial \lambda_{jj_2 a}\,\partial \lambda_{ii_1 a}}\,2\,\mu_{j_2 a} - \frac{\partial^2 h'}{\partial \mu_a\,\partial \lambda_{ii_1 a}}\,\mu_{ja} - 4\frac{\partial^2 h'}{\partial \mu_{ja}\,\partial \lambda_{ii_1 a}}\,\nu_a \stackrel{**}{=} 2\frac{\partial h'}{\partial \lambda_{a(i}}\,\delta^j_{i_1)}\,,$$

where in the passage denoted with $\stackrel{*}{=}$ we have used (11) and in that denoted with $\stackrel{**}{=}$ we have used the derivative of eq. (12) with respect to $\lambda_{ii_1 a}$. Similarly, we have

$$\frac{\partial}{\partial v_\tau^j}\frac{\partial h_a'^i}{\partial \lambda_{i_1 a}} = \frac{\partial^2 h_a'^i}{\partial \lambda_{Aa}\,\partial \lambda_{i_1 a}}\frac{\partial \lambda_{Aa}}{\partial v_\tau^j} \stackrel{*}{=} -\frac{\partial^2 h_a'^i}{\partial \lambda_a\,\partial \lambda_{i_1 a}}\,\lambda_{ja} - 2\frac{\partial^2 h_a'^i}{\partial \lambda_{j_1 a}\,\partial \lambda_{i_1 a}}\left(\lambda_{j_1 ja} + \delta_{j_1 j}\,\mu_a\right) - \frac{\partial^2 h_a'^i}{\partial \lambda_{jj_2 a}\,\partial \lambda_{i_1 a}}\,2\,\mu_{j_2 a} - \frac{\partial^2 h_a'^i}{\partial \mu_a\,\partial \lambda_{i_1 a}}\,\mu_{ja} - 4\frac{\partial^2 h_a'^i}{\partial \mu_{ja}\,\partial \lambda_{i_1 a}}\,\nu_a \stackrel{**}{=} \frac{\partial h_a'^i}{\partial \lambda_a}\,\delta_{i_1 j} + \frac{\partial h'}{\partial \lambda_{i_1 a}}\,\delta^{ij} = 2\frac{\partial h'}{\partial \lambda_{a(i}}\,\delta^j_{i_1)}\,, \tag{14}$$

where in the passage denoted with $\stackrel{*}{=}$ we have used (11), in that denoted with $\stackrel{**}{=}$ we have used the derivative of eq. (13) with respect to $\lambda_{i_1 a}$ and in the last passage we have used $(5)_1$. Consequently, we have

$$\frac{\partial}{\partial v_\tau^j}\left(\frac{\partial h'}{\partial \lambda_{ii_1 a}} - \frac{\partial h_a'^i}{\partial \lambda_{i_1 a}}\right) = 0\,.$$

This proves that $(5)_2$ doesn't depend on the reference frame.
It is interesting to see that from (14) it follows

$$\frac{\partial}{\partial v_T^j} \frac{\partial h_a'^{[i}}{\partial \lambda_{i_1]a}} = 0 \,,$$

that is, also $(5)_4$ doesn't depend on the reference frame.

- Let us now prove that $(5)_3$ is independent from the reference frame. We have

$$\frac{\partial}{\partial v_T^j} \frac{\partial h'}{\partial \mu_{ia}} = \frac{\partial^2 h'}{\partial \lambda_{Aa} \partial \mu_{ia}} \frac{\partial \lambda_{Aa}}{\partial v_T^j} \overset{*}{=} - \frac{\partial^2 h'}{\partial \lambda_a \partial \mu_{ia}} \lambda_{ja} - 2 \frac{\partial^2 h'}{\partial \lambda_{j_1 a} \partial \mu_{ia}} (\lambda_{j_1 ja} + \delta_{j_1 j} \mu_a) - \frac{\partial^2 h'}{\partial \lambda_{jj_2 a} \partial \mu_{ia}} 2 \mu_{j_2 a} - \frac{\partial^2 h'}{\partial \mu_a \partial \mu_{ia}} \mu_{ja} - 4 \frac{\partial^2 h'}{\partial \mu_{ja} \partial \mu_{ia}} \nu_a \overset{**}{=} 2 \frac{\partial h'}{\partial \lambda_{jia}} + \frac{\partial h'}{\partial \mu_a} \delta^i_j \,,$$

where in the passage denoted with $\overset{*}{=}$ we have used (11) and in that denoted with $\overset{**}{=}$ we have used the derivative of eq. (12) with respect to μ_{ia}. Similarly, we have

$$\frac{\partial}{\partial v_T^j} \frac{\partial h_a'^i}{\partial \mu_a} = \frac{\partial^2 h_a'^i}{\partial \lambda_{Aa} \partial \mu_a} \frac{\partial \lambda_{Aa}}{\partial v_T^j} \overset{*}{=} - \frac{\partial^2 h_a'^i}{\partial \lambda_a \partial \mu_a} \lambda_{ja} - 2 \frac{\partial^2 h_a'^i}{\partial \lambda_{j_1 a} \partial \mu_a} (\lambda_{j_1 ja} + \delta_{j_1 j} \mu_a) - \frac{\partial^2 h_a'^i}{\partial \lambda_{jj_2 a} \partial \mu_a} 2 \mu_{j_2 a} - \frac{\partial^2 h_a'^i}{\partial \mu_a^2} \mu_{ja} - 4 \frac{\partial^2 h_a'^i}{\partial \mu_{ja} \partial \mu_a} \nu_a \overset{**}{=} \frac{\partial h_a'^i}{\partial \lambda_a} \mu_a + \frac{\partial h'}{\partial \mu_a} \delta^{ij} = 2 \frac{\partial h'}{\partial \lambda_{jia}} + \frac{\partial h'}{\partial \mu_a} \delta^i_j \,,$$

where in the passage denoted with $\overset{*}{=}$ we have used (11), in that denoted with $\overset{**}{=}$ we have used the derivative of eq. (13) with respect to μ_a and in the last passage we have used $(5)_2$. Consequently, we have

$$\frac{\partial}{\partial v_T^j} \left(\frac{\partial h'}{\partial \mu_{ia}} - \frac{\partial h_a'^i}{\partial \mu_a} \right) = 0 \,.$$

This proves that $(5)_3$ doesn't depend on the reference frame.

- Let us prove now eq. $(6)_8$. We have

$$\frac{\partial A_{0a}^{ii_1 i_2}}{\partial v_T^j} = \frac{\partial}{\partial v_T^j} \frac{\partial h_a'^i}{\partial \lambda_{i_1 i_2 a}} = \frac{\partial^2 h_a'^i}{\partial \lambda_{Aa} \partial \lambda_{i_1 i_2 a}} \frac{\partial \lambda_{Aa}}{\partial v_T^j} \overset{*}{=} - \frac{\partial^2 h_a'^i}{\partial \lambda_a \partial \lambda_{i_1 i_2 a}} \lambda_{ja} - 2 \frac{\partial^2 h_a'^i}{\partial \lambda_{j_1 a} \partial \lambda_{i_1 i_2 a}} (\lambda_{j_1 ja} + \delta_{j_1 j} \mu_a) - \frac{\partial^2 h_a'^i}{\partial \lambda_{jj_2 a} \partial \lambda_{i_1 i_2 a}} 2 \mu_{j_2 a} - \frac{\partial^2 h_a'^i}{\partial \mu_a \partial \lambda_{i_1 i_2 a}} \mu_{ja} - 4 \frac{\partial^2 h_a'^i}{\partial \mu_{ja} \partial \lambda_{i_1 i_2 a}} \nu_a \overset{**}{=} 2 \frac{\partial h_a'^i}{\partial \lambda_{a(i_1}} \delta_{i_2)j} + \frac{\partial h'}{\partial \lambda_{ai_1 i_2}} \delta^i_j = 3 \frac{\partial h'}{\partial \lambda_{a(i_1 i_2}} \delta^{i)}_j = 3 A_{0a}^{(i_1 i_2} \delta^{i)}_j \,, \tag{15}$$

where in the passage denoted with $\overset{*}{=}$ we have used (11), in that denoted with $\overset{**}{=}$ we have used the derivative of eq. (13) with respect to $\lambda_{ai_1 i_2}$ and in the subsequent passage we have used $(5)_2$. By applying $(6)_3$ we obtain

$$\frac{\partial A_{0a}^{ii_1 i_2}}{\partial v_T^j} = 3 A_{0r}^{(i_1 i_2} \delta^{i)}_j + 6 A_{0r}^{(i_1} v_T^{i_2} \delta^{i)}_j + 3 A_{0r} v_T^{(i_1} v_T^{i_2} \delta^{i)}_j = \frac{\partial}{\partial v_T^j} \left(3 A_{0r}^{(i_1 i_2} v_T^{i)} + 3 A_{0r}^{(i_1} v_T^{i_2} v_T^{i)} + A_{0r} v_T^{i_1} v_T^{i_2} v_T^{i} \right) .$$

By integrating we obtain

$$A_{0a}^{ii_1i_2} = C^{ii_1i_2} + 3\,A_{0r}^{(i_1i_2}\,v_\tau^{i)} + 3\,A_{0r}^{(i_1}v_\tau^{i_2}\,v_\tau^{i)} + A_{0r}v_\tau^{i_1}v_\tau^{i_2}\,v_\tau^{i}\,,$$

where $C^{ii_1i_2}$ is the tensor constant, with respect to v_τ^j arising from the integration. So we can calculate the above equation in $v_\tau^j = 0$ and obtain $A_{0r}^{ii_1i_2} = C^{ii_1i_2}$ because $\left(A_{0a}^{ii_1i_2}\right)_{v_\tau^j=0} = A_{0r}^{ii_1i_2}$. By substituting this value of $C^{ii_1i_2}$ in the above expresion, we obtain $(6)_8$.
It is interesting to see that from (15) it follows

$$\frac{\partial}{\partial\,v_\tau^j}\,\frac{\partial\,h_a'^{[i}}{\partial\,\lambda_{i_1]i_2a}} = 0\,,$$

that is, also $(5)_5$ doesn't depend on the reference frame.

- Let us prove now eq. $(6)_9$. We have

$$\frac{\partial\,A_{1a}^{ii_1}}{\partial\,v_\tau^j} = \frac{\partial}{\partial\,v_\tau^j}\,\frac{\partial\,h_a'^{i}}{\partial\,\mu_{i_1a}} = \frac{\partial^2\,h_a'^{i}}{\partial\,\lambda_{Aa}\,\partial\,\mu_{i_1a}}\,\frac{\partial\,\lambda_{Aa}}{\partial\,v_\tau^j} \stackrel{*}{=} -\frac{\partial^2\,h_a'^{i}}{\partial\,\lambda_a\,\partial\,\mu_{i_1a}}\,\lambda_{ja} - \tag{16}$$
$$2\,\frac{\partial^2\,h_a'^{i}}{\partial\,\lambda_{j_1a}\,\partial\,\mu_{i_1a}}\,(\lambda_{j_1ja} + \delta_{j_1j}\,\mu_a) - \frac{\partial^2\,h_a'^{i}}{\partial\,\lambda_{jj_2a}\,\partial\,\mu_{i_1a}}\,2\,\mu_{j_2a} - \frac{\partial^2\,h_a'^{i}}{\partial\,\mu_a\,\partial\mu_{i_1a}}\,\mu_{ja} -$$
$$4\,\frac{\partial^2\,h_a'^{i}}{\partial\,\mu_{ja}\,\partial\,\mu_{i_1a}}\,\nu_a \stackrel{**}{=} 2\,\frac{\partial\,h_a'^{i}}{\partial\,\lambda_{aji_1}} + \frac{\partial\,h_a'^{i}}{\partial\,\mu_a}\,\delta_j^{i_1} + \frac{\partial\,h'}{\partial\,\mu_{ai_1}}\,\delta_j^{i} = 2\,A_{0a}^{i_1ij} + 2\,A_{1a}^{(i_1}\,\delta_j^{i)}\,,$$

where in the passage denoted with $\stackrel{*}{=}$ we have used (11), in that denoted with $\stackrel{**}{=}$ we have used the derivative of eq. (13) with respect to μ_{i_1a}. By applying $(6)_{5,7}$ we obtain

$$\frac{\partial\,A_{1a}^{ii_1}}{\partial\,v_\tau^j} = \frac{\partial}{\partial\,v_\tau^j}\Big(2\,A_{1r}^{(i}v_\tau^{i_1)} + 2\,A_{0r}^{ihi_1}v_{h\tau} + 4A_{0r}^{h(i}v_\tau^{i_1)}v_{h\tau} + A_{0r}^{ii_1}v_\tau^2 + 2\,A_{0r}^{(i}v_\tau^{i_1)}v_\tau^2 +$$
$$+ 2\,A_{0r}^{h}\,v_{h\tau}\,v_\tau^{i_1}\,v_\tau^{i} + A_{0r}\,v_\tau^2\,v_\tau^{i_1}\,v_\tau^{i} + A_{1r}\,v_\tau^{i_1}\,v_\tau^{i}\Big)\,.$$

By integrating we obtain

$$A_{1a}^{ii_1} = C^{ii_1} + 2\,A_{1r}^{(i}v_\tau^{i_1)} + 2\,A_{0r}^{ihi_1}v_{h\tau} + 4A_{0r}^{h(i}v_\tau^{i_1)}v_{h\tau} + A_{0r}^{ii_1}v_\tau^2 + 2\,A_{0r}^{(i}v_\tau^{i_1)}v_\tau^2 +$$
$$+2\,A_{0r}^{h}\,v_{h\tau}\,v_\tau^{i_1}\,v_\tau^{i} + A_{0r}\,v_\tau^2\,v_\tau^{i_1}\,v_\tau^{i} + A_{1r}\,v_\tau^{i_1}\,v_\tau^{i}\,,$$

where C^{ii_1} is the tensor constant, with respect to v_τ^j arising from the integration. So we can calculate the above equation in $v_\tau^j = 0$ and obtain $A_{1r}^{ii_1} = C^{ii_1}$ because $\left(A_{1a}^{ii_1}\right)_{v_\tau^j=0} = A_{1r}^{ii_1}$. By substituting this value of C^{ii_1} in the above expresion, we obtain $(6)_9$.
It is interesting to see that from (16) it follows

$$\frac{\partial}{\partial\,v_\tau^j}\,\frac{\partial\,h_a'^{[i}}{\partial\,\mu_{i_1]a}} = 0\,,$$

that is, also $(5)_6$ doesn't depend on the reference frame.

- Let us prove now eq. $(6)_{10}$. We have

$$\frac{\partial A^i_{2a}}{\partial v^j_\tau} = \frac{\partial}{\partial v^j_\tau}\frac{\partial h'^i_a}{\partial \nu_a} = \frac{\partial^2 h'^i_a}{\partial \lambda_{Aa}\partial \nu_a}\frac{\partial \lambda_{Aa}}{\partial v^j_\tau} \overset{*}{=} -\frac{\partial^2 h'^i_a}{\partial \lambda_a \partial \nu_a}\lambda_{ja} - 2\frac{\partial^2 h'^i_a}{\partial \lambda_{j_1 a}\partial \nu_a}(\lambda_{j_1 ja} + \delta_{j_1 j}\mu_a) - \frac{\partial^2 h'^i_a}{\partial \lambda_{jj_2 a}\partial \nu_a}2\mu_{j_2 a} - \frac{\partial^2 h'^i_a}{\partial \mu_a \partial \nu_a}\mu_{ja} - 4\frac{\partial^2 h'^i_a}{\partial \nu_a^2}\nu_a \overset{**}{=} 4\frac{\partial h'^i_a}{\partial \mu_{aj}} + \frac{\partial h'}{\partial \nu_a}\delta^i_j = 4A^{ij}_{1a} + A_{2a}\delta^i_j ,$$

where in the passage denoted with $\overset{*}{=}$ we have used (11), in that denoted with $\overset{**}{=}$ we have used the derivative of eq. (13) with respect to ν_a. By applying $(6)_{6,8}$ we obtain

$$\frac{\partial A^i_{2a}}{\partial v^j_\tau} = \frac{\partial}{\partial v^j_\tau}\Big[4A^{ih}_{1r}v_{h\tau} + 2A^i_{1r}v^2_\tau + 4A^{ihk}_{0r}v_{h\tau}v_{k\tau} + 4A^{ih}_{0r}v_{h\tau}v^2_\tau + A^i_{0r}v^4_\tau + + v^i_\tau\left(A_{2r} + 4A^h_{1r}v_{h\tau} + 2A_{1r}v^2_\tau + 4A^{hk}_{0r}v_{h\tau}v_{k\tau} + 4A^h_{0r}v_{h\tau}v^2_\tau + A_{0r}v^4_\tau\right)\Big] .$$

By integrating we obtain

$$A^i_{2a} = C^i + 4A^{ih}_{1r}v_{h\tau} + 2A^i_{1r}v^2_\tau + 4A^{ihk}_{0r}v_{h\tau}v_{k\tau} + 4A^{ih}_{0r}v_{h\tau}v^2_\tau + A^i_{0r}v^4_\tau + + v^i_\tau\left(A_{2r} + 4A^h_{1r}v_{h\tau} + 2A_{1r}v^2_\tau + 4A^{hk}_{0r}v_{h\tau}v_{k\tau} + 4A^h_{0r}v_{h\tau}v^2_\tau + A_{0r}v^4_\tau\right) ,$$

where C^i is the tensor constant, with respect to v^j_τ arising from the integration. So we can calculate the above equation in $v^j_\tau = 0$ and obtain $A^i_{2r} = C^i$ because $\left(A^i_{2a}\right)_{v^j_\tau=0} = A^i_{2r}$. By substituting this value of C^i in the above expresion, we obtain $(6)_{10}$.

- Let us prove now eq. $(6)_{11}$. We have

$$\frac{\partial H^{ill}_{Va}}{\partial v^j_\tau} = \frac{\partial}{\partial v^j_\tau}\frac{\partial h'^i_a}{\partial \mu_{Va}} = \frac{\partial^2 h'^i_a}{\partial \lambda_{Aa}\partial \mu_{Va}}\frac{\partial \lambda_{Aa}}{\partial v^j_\tau} \overset{*}{=} -\frac{\partial^2 h'^i_a}{\partial \lambda_a \partial \mu_{Va}}\lambda_{ja} - 2\frac{\partial^2 h'^i_a}{\partial \lambda_{j_1 a}\partial \mu_{Va}}(\lambda_{j_1 ja} + \delta_{j_1 j}\mu_a) - \frac{\partial^2 h'^i_a}{\partial \lambda_{jj_2 a}\partial \mu_{Va}}2\mu_{j_2 a} - \frac{\partial^2 h'^i_a}{\partial \mu_a \partial \nu_a}\mu_{ja} - 4\frac{\partial^2 h'^i_a}{\partial \nu_a \partial \mu_{Va}}\nu_a \overset{**}{=} \frac{\partial h'_a}{\partial \mu_{Va}}\delta^i_j ,$$

where in the passage denoted with $\overset{*}{=}$ we have used (11), in that denoted with $\overset{**}{=}$ we have used the derivative of eq. (13) with respect to ν_a. So we have obtained

$$\frac{\partial H^{ill}_{Va}}{\partial v^j_\tau} = \frac{\partial}{\partial v^j_\tau}\left(\frac{\partial h'_a}{\partial \mu_{Va}}v^i_\tau\right) = \frac{\partial}{\partial v^j_\tau}\left(\frac{\partial h'}{\partial \mu_{Vr}}v^i_\tau\right) = \frac{\partial}{\partial v^j_\tau}\left(H^{ll}_{Vr}v^i_\tau\right) .$$

By integrating we obtain

$$H^{ill}_{Va} = C^i + H^{ll}_{Vr}v^i_\tau ,$$

where C^i is the tensor constant, with respect to v^j_τ arising from the integration. So we can calculate the above equation in $v^j_\tau = 0$ and obtain $H^{ill}_{Vr} = C^i$ because $\left(H^{ill}_{Va}\right)_{v^j_\tau=0} = H^{ill}_{Vr}$. By substituting this value of C^i in the above expresion, we obtain $(6)_{11}$.

2.1 An historical and useful change of variables

Starting from the independent variables A_0, A_0^i, A_0^{ij}, A_1, A_1^i, A_2, H_V^{ill}, it is useful to define A_{0I}, v^i, A_{0I}^{ij}, A_{1I}, A_{1I}^i, A_{2I}, H_{VI}^{ill} from the relations

$$\begin{aligned}
&A_0 = A_{0I}\,,\ A_0^i = A_{0I}v^i\,,\ A_0^{ij} = A_{0I}^{ij} + A_{0I}v^iv^j\,,\\
&A_1 = A_{1I} + A_{0I}v^2\,,\\
&A_1^i = A_{1I}^i + A_{1I}v^i + 2A_{0I}^{hi}v_h + A_{0I}^i v^2 + A_{0I}v^2v^i\,,\\
&A_2 = A_{2I} + 4A_{1I}^h v_h + 2A_{1I}v^2 + 4A_{0I}^{hk}v_hv_k + A_{0r}v^4\,,\\
&H_V^{ll} = H_{VI}^{ll}\,.
\end{aligned} \tag{17}$$

Here v^i is the velocity of the fluid. Formally this system can be obtained from eqs. (6) omitting the suffix "a", replacing the suffix "r" with I, replacing v_τ^i with v^i and putting $A_{0I}^i = 0$. A similar transformation is considered from the dependent variables $A_0^{ii_1i_2}$, $A_1^{ii_1}$, A_2^i to $A_{0I}^{ii_1i_2}$, $A_{1I}^{ii_1}$, A_{2I}^i according to the law

$$\begin{aligned}
&A_0^{ii_1i_2} - v^iA_0^{i_1i_2} = A_{0I}^{ii_1i_2} + 2A_{0I}^{i(i_1}v^{i_2)}\,,\\
&A_1^{ii_1} - v^iA_1^{i_1} = A_{1I}^{ii_1} + A_{1I}^i v^{i_1} + 2\,A_{0I}^{ihi_1}v_h + 2A_{0I}^{ih}v_hv^{i_1} + A_{0I}^{ii_1}v^2\,,\\
&A_2^i - v^iA_2 = A_{2I}^i + 4A_{1I}^{ih}v_h + 4A_{0I}^{ihk}v_hv_k + 4A_{0I}^{ih}v_hv^2\,,\\
&H_V^{ill} - v^iH_V^{ll} = H_{VI}^{ill}\,.
\end{aligned} \tag{18}$$

Since the above considerations don't depend on the physical menaning of v_τ^i, they hold also with the new variables. In particular, we have that the independent variables A_{0I}, A_{0I}^{ij}, A_{1I}, A_{1I}^i, A_{2I}, H_{VI}^{ll} and the dependent variables $A_{0I}^{ii_1i_2}$, $A_{1I}^{ii_1}$, A_{2I}^i, H_{VI}^{ill} don' t depend on the reference frame, while only for v^i we have that $v_a^i = v_r^i + v_\tau^i$. This means that $A_{0I}^{ii_1i_2}$, $A_{1I}^{ii_1}$, A_{2I}^i, H_{VI}^{ill} are functions only of A_{0I}, A_{0I}^{ij}, A_{1I}, A_{1I}^i, A_{2I}, H_{VI}^{ll} . They don' t depend on v^i. After that, the dependence of $A_0^{ii_1i_2}$, $A_1^{ii_1}$, A_2^i, H_V^{ill} on the velocity v^i is simply dictated by eqs. (18). So the problem of finding the closure is a little simplified. Now the problem is how to express this situation in terms of the Lagrange multipliers. In any case, also with the previous variables, we had to find the Lagramge multipiers in terms of A_0, A_0^i, A_0^{ij}, A_1, A_1^i, A_2, H_{VI}^{ll} (which we now call simply F^A) inverting the relations $F^A = F^A(\lambda_B)$ and, after that, to substitute them in $F^{iA} = F^{iA}(\lambda_B)$.

So nothing changes if we consider the expressions (17) for F^A and the expressions (18) for F^{iA}. More than that, if we want to find only the expressions of $A_{0I}^{ii_1i_2}$, $A_{1I}^{ii_1}$, A_{2I}^i it suffices to calculate (17) and (18) in $v^i = 0$. This means that we have to calculate the Lagrange multipliers from the relations

$$A_{0I} = \frac{\partial h'}{\partial \lambda},\ 0 = \frac{\partial h'}{\partial \lambda_{i_1}},\ A_{0I}^{i_1i_2} = \frac{\partial h'}{\partial \lambda_{i_1i_2}},\ A_{1I} = \frac{\partial h'}{\partial \mu},\ A_{1I}^{i_1} = \frac{\partial h'}{\partial \mu_{i_1}},\ A_{2I} = \frac{\partial h'}{\partial \nu},\ H_{VI}^{ll} = \frac{\partial h'}{\partial \mu_V}\,. \tag{19}$$

After that, we have to substitute them in

$$A_{0I}^{ii_1i_2} = \frac{\partial h_I'^i}{\partial \lambda_{i_1i_2}}\,,\ A_{1I}^{ii_1} = \frac{\partial h_I'^i}{\partial \mu_{i_1}}\,,\ A_{2I}^i = \frac{\partial h_I'^i}{\partial \nu}\,,\ H_{VI}^{ill} = \frac{\partial h_I'^i}{\partial \mu_V}\,. \tag{20}$$

More particulars for this procedure can be found in the article by Pennisi and Ruggeri [11]. To find the explicit closure of (1) both at equilibrium that in the first deviation from equilibrium some integrals are necessary; I report them in the next section.

III. Some Integrals Necessary To Close The System

The following tensors are expressed in terms of

$$f = e^{-1-\frac{m}{k_B}\lambda_I} e^{-\frac{\mathcal{I}^R+\mathcal{I}^V}{k_B T}} e^{-\frac{m}{2k_B T}\xi^2}$$

and are necessary to find the closure of the field equations (1):

$$\begin{aligned}
A_{0E} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,\\
A_{0IE}^{i_1} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^{i_1}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,\\
A_{0IE}^{i_1 i_2} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^{i_1}\,\xi^{i_2}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,\\
A_{1IE} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,\\
A_{1IE}^{i_1} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\xi^{i_1}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,\\
A_{2IE} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,\\
H_{VIE}^{ll} &= 2\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\mathcal{I}^V\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,
\end{aligned}\tag{21}$$

$$\begin{aligned}
A_{0IE}^{ii_1i_2} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^{i}\,\xi^{i_1}\,\xi^{i_2}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,\\
A_{1IE}^{ii_1} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\xi^{i}\,\xi^{i_1}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,\\
A_{2IE}^{i} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\xi^{i}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,\\
H_{VIE}^{ill} &= 2\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^{i}\,\mathcal{I}^V\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,.
\end{aligned}\tag{22}$$

$$\begin{aligned}
B^{i_1i_2j_1j_2} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^{i_1}\,\xi^{i_2}\,\xi^{j_1}\,\xi^{j_2}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,\\
B^{i_1i_2j} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^{i_1}\,\xi^{i_2}\,\xi^{j}\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,\\
B_2^{i_1i_2} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^{i_1}\,\xi^{i_2}\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,\\
B_V^{i_1i_2} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^{i_1}\,\xi^{i_2}\,\frac{2\mathcal{I}^V}{m}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,\\
B_1^{j} &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^{j}\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,\\
B_1 &= m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\left(\xi^2+\frac{2\mathcal{I}}{m}\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,
\end{aligned}$$

$$B_2^j = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^j\left(\xi^2+\frac{2\mathcal{I}}{m}\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,$$
$$B_2 = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,$$
$$B_{1V} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\frac{2\mathcal{I}^V}{m}\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,$$
$$B_3^{ij} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^j\left(\xi^2+\frac{2\mathcal{I}}{m}\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,$$
$$B_3^i = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,$$
$$B_V^i = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\frac{2\mathcal{I}^V}{m}\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,$$
$$B_3 = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,$$
$$B_{2V} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\frac{2\mathcal{I}^V}{m}\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}$$
$$B_{3V} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\left(\frac{2\mathcal{I}^V}{m}\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,$$

$$B^{ii_1i_2j_1j_2} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^{i_1}\,\xi^{i_2}\,\xi^{j_1}\,\xi^{j_2}\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,$$
$$B_2^{ii_1i_2j} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^{i_1}\,\xi^{i_2}\,\xi^j\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,$$
$$B_2^{ii_1i_2} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^{i_1}\,\xi^{i_2}\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,$$
$$B_V^{ii_1i_2} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^{i_1}\,\xi^{i_2}\,\frac{2\mathcal{I}^V}{m}\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,$$
$$B_3^{ii_1j} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^{i_1}\,\xi^j\left(\xi^2+\frac{2\mathcal{I}}{m}\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,$$
$$B_4^{ij} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^j\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}\,,$$
$$B_{2V}^{ij} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\,\xi^j\left(\xi^2+\frac{2\mathcal{I}}{m}\right)\frac{2\mathcal{I}^V}{m}\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}\,,$$
$$B_4^i = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi}=0\,,$$
$$B_{2V}^i = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\left(\xi^4+\frac{4\mathcal{I}}{m}\xi^2\right)\frac{2\mathcal{I}^V}{m}\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,$$
$$B_{3V}^i = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^i\left(\frac{2\mathcal{I}^V}{m}\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^V\,d\vec{\xi}=0\,,$$

where we have already put equal to zero the integrals with an odd number of ξ^i because this property comes out from the representation theorems.

So we need to find the expressions of the remaining integrals. Their calculations have in common some aspects that can be summarized in

$$
\begin{aligned}
&\int_0^{+\infty} e^{-\frac{m}{2k_BT}s^2} s^{2\beta}\, ds = \frac{1}{2}\left(\frac{2k_BT}{m}\right)^{\beta+\frac{1}{2}} \Gamma\left(\beta+\frac{1}{2}\right) = \\
&\qquad = \frac{1}{2}\left(\frac{2k_BT}{m}\right)^{\beta+\frac{1}{2}} \frac{1}{2^\beta}\frac{(2\beta+1)!!}{2\beta+1}\sqrt{\pi}\,, \\
&A_R(T) \stackrel{def.}{=} \int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_BT}} \varphi(\mathcal{I}^R)\, d\mathcal{I}^R, \quad A_V(T) \stackrel{def.}{=} \int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_BT}} \psi(\mathcal{I}^V)\, d\mathcal{I}^V, \\
&B_R(T) \stackrel{def.}{=} \int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_BT}} \mathcal{I}^R \varphi(\mathcal{I}^R)\, d\mathcal{I}^R = k_BT^2 \frac{\partial A_R(T)}{\partial T}, \\
&B_V(T) \stackrel{def.}{=} \int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_BT}} \mathcal{I}^V \psi(\mathcal{I}^V)\, d\mathcal{I}^V = k_BT^2 \frac{\partial A_V(T)}{\partial T}, \\
&C_R(T) \stackrel{def.}{=} \int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_BT}} \left(\mathcal{I}^R\right)^2 \varphi(\mathcal{I}^R)\, d\mathcal{I}^R = k_BT^2 \frac{\partial B_R(T)}{\partial T}, \\
&C_V(T) \stackrel{def.}{=} \int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_BT}} \left(\mathcal{I}^V\right)^2 \psi(\mathcal{I}^V)\, d\mathcal{I}^V = k_BT^2 \frac{\partial B_V(T)}{\partial T}, \\
&\beta_R(T) \stackrel{def.}{=} \frac{B_R(T)}{A_R(T)}\frac{1}{k_BT} = T\frac{\partial}{\partial T}\ln A_R(T), \quad \beta_V(T) \stackrel{def.}{=} \frac{B_V(T)}{A_V(T)}\frac{1}{k_BT} = T\frac{\partial}{\partial T}\ln A_V(T) \\
&\gamma_R(T) \stackrel{def.}{=} \frac{C_R(T)}{A_R(T)}\left(\frac{1}{k_BT}\right)^2 = \left(T\frac{\partial}{\partial T}\ln A_R(T)\right)\left(T\frac{\partial}{\partial T}\ln B_R(T)\right), \\
&\gamma_V(T) \stackrel{def.}{=} \frac{C_V(T)}{A_V(T)}\left(\frac{1}{k_BT}\right)^2 = \left(T\frac{\partial}{\partial T}\ln A_V(T)\right)\left(T\frac{\partial}{\partial T}\ln B_V(T)\right).
\end{aligned}
\tag{23}
$$

In the first one of these β is an arbitrary integer number, Γ is the Gamma function and the integration is performed with the substitution $s = \sqrt{\frac{2k_BT}{m}}\,x$; finally, in the last passage, we have used a property of the Gamma function.
Thanks to these preliminary aspects, we can calculate the above integrals and the results are:

$$
\begin{aligned}
&A_{0E} \stackrel{def.}{=} \rho \quad \text{and can be obtained from} \quad e^{-1-\frac{m}{k_B}\lambda_{IE}} = \frac{\rho}{m}\left(\frac{m}{2k_BT\pi}\right)^{\frac{3}{2}} \frac{1}{A_R(T)A_V(T)}, \\
&A^{ij}_{0IE} = p\,\delta^{ij} \quad \text{with}\, p = \rho\frac{k_BT}{m}, \\
&A_{1E} = 2\rho\epsilon \quad \text{with}\, \epsilon = \frac{p}{\rho}\left[\frac{3}{2} + T\frac{\partial}{\partial T}\ln\Big(A_R(T)A_V(T)\Big)\right], \\
&A_{2IE} = 3\rho\left(\frac{p}{\rho}\right)^2\left[5 + 4T\frac{\partial}{\partial T}\ln\Big(A_R(T)A_V(T)\Big)\right], \\
&H^{ll}_{VIE} = 2pT\frac{\partial}{\partial T}\ln\Big(A_V(T)\Big), \\
&A^{ij}_{1IE} = \psi\,\delta^{ij} \quad \text{with} \quad \psi = \rho\left(\frac{p}{\rho}\right)^2\left[5 + 2T\frac{\partial}{\partial T}\ln\Big(A_R(T)A_V(T)\Big)\right],
\end{aligned}
$$

$$B^{i_1 i_2 j_1 j_2} = 3\,\rho \left(\frac{p}{\rho}\right)^2 \delta^{(i_1 i_2}\delta^{j_1 j_2)}\,,$$
$$B_2^{i_1 i_2} = \psi_1\, \delta^{i_1 i_2} \quad , \quad \text{with}$$
$$\psi_1 = 5\,\rho \left(\frac{p}{\rho}\right)^3 \left[7 + 4T\,\frac{\partial}{\partial T}\,\ln\Big(A_R(T)\,A_V(T)\Big)\right],$$
$$B_V^{i_1 i_2} = \psi_2\, \delta^{i_1 i_2} \quad , \quad \text{with} \quad \psi_2 = 2\rho \left(\frac{p}{\rho}\right)^2 \left(T\,\frac{\partial}{\partial T}\,\ln A_V(T)\right),$$

$$B_1 = \rho \left(\frac{p}{\rho}\right)^2 \cdot$$
$$\cdot \left[15 + 12T\,\frac{\partial}{\partial T}\,\ln\Big(A_R(T)\,A_V(T)\Big) + 4\Big(\gamma_V(T) + \gamma_R(T) + 2\,\beta_V(T)\beta_R(T)\Big)\right],$$

$$B_2 = 3\,\rho \left(\frac{p}{\rho}\right)^3 \cdot$$
$$\cdot \left[35 + 30T\,\frac{\partial}{\partial T}\,\ln\Big(A_R(T)\,A_V(T)\Big) + 8\Big(\gamma_V(T) + \gamma_R(T) + 2\,\beta_V(T)\beta_R(T)\Big)\right],$$

$$B_{1V} = \rho \left(\frac{p}{\rho}\right)^2 \Big(4\,\gamma_V(T) + 6\,\beta_V(T) + 4\,\beta_R(T)\,\beta_V(T)\Big)$$

$$B_3^{ij} = \psi_3\, \delta^{ij} \quad , \quad \text{with} \quad \psi_3 = \rho \left(\frac{p}{\rho}\right)^3 \cdot$$
$$\cdot \left[35 + 20\,T\,\frac{\partial}{\partial T}\,\ln\Big(A_R(T)\,A_V(T)\Big) + 4\Big(\gamma_V(T) + \gamma_R(T) + 2\,\beta_V(T)\beta_R(T)\Big)\right],$$

$$B_3 = 15\,\rho \left(\frac{p}{\rho}\right)^4 \cdot$$
$$\cdot \left[63 + 56\,T\,\frac{\partial}{\partial T}\,\ln\Big(A_R(T)\,A_V(T)\Big) + 16\Big(\gamma_V(T) + \gamma_R(T) + 2\,\beta_V(T)\beta_R(T)\Big)\right],$$

$$B_{2V} = 6\,\rho \left(\frac{p}{\rho}\right)^3 \left[5\,T\,\frac{\partial}{\partial T}\,\ln\Big(A_V(T)\Big) + 4\Big(\gamma_V(T) + \beta_V(T)\beta_R(T)\Big)\right],$$

$$B_{3V} = 4\,\rho \left(\frac{p}{\rho}\right)^2 \gamma_V(T)\,,$$

$$B_2^{i_1 i_2 j_1 j_2} = \psi_4\, \delta^{(i_1 i_2}\delta^{j_1 j_2)} \quad , \quad \text{with}$$
$$\psi_4 = 3\,\rho \left(\frac{p}{\rho}\right)^3 \left[7 + 2\,T\,\frac{\partial}{\partial T}\,\ln\Big(A_R(T)\,A_V(T)\Big)\right],$$

$$B_4^{ij} = \psi_5 \, \delta^{ij} \quad , \quad \text{with} \quad \psi_5 = 5 \, \rho \left(\frac{p}{\rho} \right)^4 \cdot$$

$$\cdot \left[63 + 42 \, T \, \frac{\partial}{\partial T} \ln \Big(A_R(T) \, A_V(T) \Big) + 8 \Big(\gamma_V(T) + \gamma_R(T) + 2 \, \beta_V(T) \beta_R(T) \Big) \right] ,$$

$$B_{2V}^{ij} == \psi_6 \, \delta^{ij} \quad , \quad \text{with}$$

$$\psi_6 = 2 \, \rho \left(\frac{p}{\rho} \right)^3 \left[5 \, T \, \frac{\partial}{\partial T} \ln \Big(A_V(T) \Big) + 2 \Big(\gamma_V(T) + \beta_V(T) \beta_R(T) \Big) \right] ,$$

The others can be calculated in a similar way:

- Let us begin with A_{0E}.

For integrating in $d\vec{\xi}$ we can use the spherical coordinates $\hat{\xi}^1 = s \sin\theta \cos\varphi$, $\hat{\xi}^2 = s \sin\theta \sin\varphi$, $\hat{\xi}^3 = s \cos\theta$, with $\theta \in [0\,,\,\pi[$, $\varphi \in [0\,,\,2\,\pi[$, $s \in [0\,,\,+\infty[$, whose Jacobian is $J = s^2 \sin\theta$.
After that, we have

$$A_{0E} = \rho = 4\pi \, m \, e^{-1 - \frac{m}{k_B} \lambda_I} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^2 \, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \tag{24}$$

$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 4\pi \, m \, e^{-1 - \frac{m}{k_B} \lambda_I} \, \frac{1}{2} \left(\frac{2 \, k_B T}{m} \right)^{\frac{3}{2}} \frac{1}{2} \sqrt{\pi} \, A_R(T) A_V(T) \, ,$$

from which the above written expression follows.

- Let us proceed with A_{0IE}^{ij}.

From the Representation Theorems, we have $A_{0IE}^{ij} = p \, \delta^{ij}$ whose trace is

$$3 \, p = m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f \, \xi^2 \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^V \, d\vec{\xi} =$$

$$= 4\pi \, m \, e^{-1 - \frac{m}{k_B} \lambda_I} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^4 \, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right)$$

$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 4\pi \, m \, e^{-1 - \frac{m}{k_B} \lambda_I} \, \frac{1}{2} \left(\frac{2 \, k_B T}{m} \right)^{\frac{5}{2}} \frac{3}{4} \sqrt{\pi} \, A_R(T) A_V(T) \, ,$$

which can be divided by (24) and gives $p = \rho \, \frac{k_B T}{m}$ as reported above.

- Let us proceed with H_{VIE}^{ll}.

It is similar to A_{0E} but there is the further factor $\frac{2 \mathcal{I}^V}{m}$. So we obtain

$$H_{VIE}^{ll} = 4\pi \, e^{-1 - \frac{m}{k_B} \lambda_I} \left(\frac{2 \, k_B T}{m} \right)^{\frac{3}{2}} \frac{1}{2} \sqrt{\pi} \, A_R(T) B_V(T) \, ,$$

which can be divided by (24) and gives

$$H_{VIE}^{ll} = \frac{2}{m} \frac{B_V}{AV} = 2 \, \frac{k_B \, T}{m} T \, \frac{\partial}{\partial T} \ln A_V(T) \, ,$$

i.e., the above reported expression.

- Let us proceed with A_{1IE}.

It can be rewritten as $A_{1IE} = 3p + H^{ll}_{VIE} + H^{ll}_{RIE}$ where H^{ll}_{RIE} is similar to H^{ll}_{VIE} but with R instead of V. So we can say that

$$A_{1IE} = 3p + 2pT\frac{\partial}{\partial T}\ln A_V(T) + 2pT\frac{\partial}{\partial T}\ln A_R(T),$$

i.e., the above reported expression.

- Let us proceed with A_{2IE}.

We have

$$A_{2IE} = 4\pi m\, e^{-1-\frac{m}{k_B}\lambda_I}\left(\int_0^{+\infty} e^{-\frac{m}{2k_BT}s^2} s^6\, ds\right)\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_BT}}\varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right)$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_BT}}\psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) + 16\pi\, e^{-1-\frac{m}{k_B}\lambda_I}\left(\int_0^{+\infty} e^{-\frac{m}{2k_BT}s^2} s^4\, ds\right)$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_BT}}\mathcal{I}^R\varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right)\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_BT}}\psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) +$$
$$+16\pi\, e^{-1-\frac{m}{k_B}\lambda_I}\left(\int_0^{+\infty} e^{-\frac{m}{2k_BT}s^2} s^4\, ds\right)\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_BT}}\varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right)$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_BT}}\mathcal{I}^v\psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) = 4\pi m\, e^{-1-\frac{m}{k_B}\lambda_I}\left[\frac{1}{2}\left(\frac{2k_BT}{m}\right)^{\frac{7}{2}}\frac{15}{8}\sqrt{\pi}\, A_R(T)A_V(T) + \right.$$
$$\left. + \frac{4}{m}\frac{1}{2}\left(\frac{2k_BT}{m}\right)^{\frac{5}{2}}\frac{3}{4}\sqrt{\pi}\,(B_RA_V + B_VA_R)\right],$$

which can be divided by (24) and gives

$$\frac{A_{2IE}}{\rho} = \left(\frac{k_BT}{m}\right)^2\left[15 + \frac{12}{k_BT}\left(\frac{B_R}{A_R} + \frac{B_V}{A_V}\right)\right],$$

from which the above reported expression follows.

- Let us proceed with A^{ij}_{1IE}.

From the Representation Theorems, we have $A^{ij}_{1IE} = \psi\,\delta^{ij}$ whose trace is

$$3\psi = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\left(\xi^2 + \frac{2\mathcal{I}}{m}\right)\xi^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^V\, d\vec{\xi}.$$

This is similar to A_{2IE} except that we have now the factor $\frac{2\mathcal{I}}{m}$ instead of $\frac{4\mathcal{I}}{m}$. So we have

$$3\frac{\psi}{\rho} = \left(\frac{k_BT}{m}\right)^2\left[15 + \frac{6}{k_BT}\left(\frac{B_R}{A_R} + \frac{B_V}{A_V}\right)\right],$$

from which the above reported expression follows.

- Let us proceed with $B^{i_1i_2j_1j_2}$.

From the Representation Theorems, we have $B^{i_1 i_2 j_1 j_2} = b\, \delta^{(i_1 i_2} \delta^{j_1 j_2)}$ whose double trace is

$$5\, b = 4\, \pi\, m\, e^{-1-\frac{m}{k_B} \lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^6\, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R)\, d\mathcal{I}^R \right)$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V)\, d\mathcal{I}^V \right) = 4\, \pi\, m\, e^{-1-\frac{m}{k_B} \lambda_{IE}} \frac{1}{2} \left(\frac{2\, k_B\, T}{m} \right)^{\frac{7}{2}} \frac{15}{8} \sqrt{\pi}\, A_R A_V\, ,$$

which can be divided by (24) and gives $b = 3\, \rho \left(\frac{p}{\rho} \right)^2$ from which the above reported expression follows.

- Similarly, $B_2^{i_1 i_2 j_1 j_2} = \psi_4\, \delta^{(i_1 i_2} \delta^{j_1 j_2)}$,

whose double trace is

$$5\, \psi_4 = 4\, \pi\, m\, e^{-1-\frac{m}{k_B} \lambda_{IE}} \left[\left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^8\, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R)\, d\mathcal{I} \right) \right.$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V)\, d\mathcal{I} \right) + \frac{2}{m} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^6\, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \mathcal{I}^R \varphi(\mathcal{I}^R)\, d\mathcal{I}^R \right)$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V)\, d\mathcal{I} \right) + \frac{2}{m} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^6\, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R)\, d\mathcal{I}^R \right)$$
$$\left. \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \mathcal{I}^V \psi(\mathcal{I}^V)\, d\mathcal{I} \right) \right] =$$
$$= 4\, \pi\, m\, e^{-1-\frac{m}{k_B} \lambda_{IE}} \sqrt{\pi} \frac{15}{32} \left(\frac{2\, k_B\, T}{m} \right)^{\frac{9}{2}} A_R\, A_V \left[7 + \frac{2}{k_B\, T} \left(\frac{B_R}{A_R} + \frac{B_V}{A_V} \right) \right] ,$$

which can be divided by (24) and gives

$$\psi_4 = 3\, \rho \left(\frac{k_B\, T}{m} \right)^3 \left[7 + \frac{2}{k_B\, T} \left(\frac{B_R}{A_R} + \frac{B_V}{A_V} \right) \right] ,$$

from which the above reported expression follows.

- By proceeding in this way, we have $B_2^{i_1 i_2} = \psi_1\, \delta^{i_1 i_2}$, whose trace is

$$3\, \psi_1 = 4\, \pi\, m\, e^{-1-\frac{m}{k_B} \lambda_{IE}} \left[\left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^8\, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R)\, d\mathcal{I}^R \right) + \right.$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V)\, d\mathcal{I}^V \right) + \frac{4}{m} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^6\, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \mathcal{I}^R \varphi(\mathcal{I}^R)\, d\mathcal{I}^R \right)$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V)\, d\mathcal{I}^V \right) + \frac{4}{m} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^6\, d\, s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R)\, d\mathcal{I}^R \right)$$
$$\left. \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V) \mathcal{I}^V\, d\mathcal{I}^V \right) \right] =$$
$$= 4\, \pi\, m\, e^{-1-\frac{m}{k_B} \lambda_{IE}} \left[7 + \frac{4}{k_B\, T} \left(\frac{B_R}{A_R} + \frac{B_V}{A_V} \right) \right] \frac{15}{32} \left(\frac{2\, k_B\, T}{m} \right)^{\frac{9}{2}} \sqrt{\pi} A_R\, A_V\, ,$$

which can be divided by (24) and gives

$$\psi_1 = 5\, \rho \left(\frac{k_B\, T}{m} \right)^3 \left[7 + \frac{4}{k_B\, T} \left(\frac{B_R}{A_R} + \frac{B_V}{A_V} \right) \right] ,$$

from which the above reported expression follows.

- Let us consider now $B_V^{ij} = \psi_2 \, \delta^{ij}$.

We can repeat the same passages done for A_{0IE}^{ij} paying attention to the fact that now there is the extra factor $2 \frac{\mathcal{I}^V}{m}$. So we obtain

$$3\,\psi_2 = 8\pi \, e^{-1-\frac{m}{k_B}\lambda_I} \, \frac{1}{2} \left(\frac{2\,k_B T}{m} \right)^{\frac{5}{2}} \frac{3}{4} \sqrt{\pi} \, A_R(T) B_V(T) \, ,$$

which can be divided by (24) and gives $\psi_2 = 2\,\rho \left(\frac{p}{\rho}\right)^2 \beta_V$ from which it folows the expression reported above.

- Let us consider now $B_{1V} = B_{11V} + B_{21V} + B_{31V}$ where

$$B_{11V} = m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \xi^2 \, \frac{2\,\mathcal{I}^V}{m} \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\,\mathcal{I}^R \, d\,\mathcal{I}^v \, d\vec{\xi} \, ,$$

$$B_{21V} = \frac{4}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, (\mathcal{I}^V)^2 \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\,\mathcal{I}^R \, d\,\mathcal{I}^v \, d\vec{\xi} \, ,$$

$$B_{31V} = \frac{4}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \mathcal{I}^R \mathcal{I}^V \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\,\mathcal{I}^R \, d\,\mathcal{I}^v \, d\vec{\xi} \, ,$$

The first one of these scalars is similar to A_{0IE}^{ll} but there is the extra factor $2 \frac{\mathcal{I}^V}{m}$. So we desume that

$$B_{11V} = \frac{8}{m} \pi \, m \, e^{-1-\frac{m}{k_B}\lambda_I} \, \frac{1}{2} \left(\frac{2\,k_B T}{m} \right)^{\frac{5}{2}} \frac{3}{4} \sqrt{\pi} \, A_R(T) B_V(T) \, ,$$

which can be divided by (24) and gives $B_{11V} = 6\,\rho \left(\frac{p}{\rho}\right)^2 \beta_V$.

The second one of these scalars is similar to A_{0E} but there is the extra factor $\left(2 \frac{\mathcal{I}^V}{m}\right)^2$. So we desume that

$$B_{21V} = \frac{16}{m} \pi \, e^{-1-\frac{m}{k_B}\lambda_I} \, \frac{1}{2} \left(\frac{2\,k_B T}{m} \right)^{\frac{3}{2}} \frac{1}{2} \sqrt{\pi} \, A_R(T) C_V(T) = \frac{4\,\rho}{m^2} \frac{C_V}{A_V} = 4\,\rho \left(\frac{p}{\rho} \right)^2 \gamma_V \, .$$

The third scalar is similar to A_{0E} but there is the extra factor $4 \frac{\mathcal{I}^R \mathcal{I}^V}{m^2}$. So we desume that

$$B_{31V} = \frac{16}{m} \pi \, e^{-1-\frac{m}{k_B}\lambda_I} \, \frac{1}{2} \left(\frac{2\,k_B T}{m} \right)^{\frac{3}{2}} \frac{1}{2} \sqrt{\pi} \, B_R(T) B_V(T) = \frac{4\,\rho}{m^2} \frac{B_R}{A_R} \frac{B_V}{A_V} =$$

$$= 4\,\rho \left(\frac{p}{\rho} \right)^2 \beta_R \, \beta_V \, ,$$

- Let us consider now B_3^{ij}.

We note that $B_3^{ij} = B_2^{ij} + B^{*ij}_3$ with

$$B^{*ij}_3 = \frac{4}{m} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi_E}{k_B}} \xi^i \xi^j \mathcal{I}^2 \phi(\mathcal{I}) \, d\vec{\xi} \, d\mathcal{I} = b_4 \, \delta^{ij} ,$$

whose trace is

$$3 \, b_4 = \frac{4}{m} 4 \pi \, m \, e^{-1-\frac{m}{k_B} \lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2 k_B T} s^2} s^4 \, d\, s \right) \left[\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \left(\mathcal{I}^R\right)^2 \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) + \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \left(\mathcal{I}^V\right)^2 \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) + 2 \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \mathcal{I}^R \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \mathcal{I}^V \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) \right] =$$

$$= 4 \pi \, m \, e^{-1-\frac{m}{k_B} \lambda_{IE}} \, \frac{3}{2 \, m} \left(\frac{2 \, k_B \, T}{m} \right)^{\frac{5}{2}} \sqrt{\pi} \left(C_R \, A_V + A_R \, C_V + 2 \, B_R \, B_V \right),$$

which can be divided by (24) and gives

$$b_4 = 4 \, \rho \left(\frac{k_B \, T}{m} \right)^3 \left(\frac{1}{k_B \, T} \right)^2 \left(\frac{C_R}{A_R} + \frac{C_V}{A_V} + 2 \, \frac{B_R}{A_R} \frac{B_V}{A_V} \right) .$$

By using this result, we have $B_3^{ij} = (\psi_1 + b_4) \, \delta^{ij} = \psi_3 \, \delta^{ij}$ with

$$\psi_3 = \psi_1 + b_4 = \rho \left(\frac{k_B \, T}{m} \right)^3 \left[35 + \frac{20}{k_B \, T} \left(\frac{B_R}{A_R} + \frac{B_V}{A_V} \right) + \left(\frac{1}{k_B \, T} \right)^2 \left(4 \, \frac{C_R}{A_R} + 4 \, \frac{C_V}{A_V} + 8 \, \frac{B_R}{A_R} \frac{B_V}{A_V} \right) \right] ,$$

from which the above reported expression follows.

- Let us consider now $B_4^{ij} = \psi_5 \, \delta^{ij}$.

It can be written as $B_4^{ij} = B_{14}^{ij} + B_{24}^{ij} + B_{34}^{ij} + B_{44}^{ij} + B_{54}^{ij} + B_{64}^{ij}$ with

$$B_{14}^{ij} = m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B} \chi_E} \xi^i \xi^j \xi^6 \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{24}^{ij} = m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B} \chi_E} \xi^i \xi^j \xi^4 \, \frac{6 \, \mathcal{I}^R}{m} \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{34}^{ij} = m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B} \chi_E} \xi^i \xi^j \xi^4 \, \frac{6 \, \mathcal{I}^V}{m} \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{44}^{ij} = 8 \, m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B} \chi_E} \xi^i \xi^j \xi^2 \left(\frac{\mathcal{I}^R}{m} \right)^2 \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B^{ij}_{54} = 8\,m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\, \xi^i\, \xi^j\, \xi^2 \left(\frac{\mathcal{I}^V}{m}\right)^2 \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^v\, d\vec{\xi},$$

$$B^{ij}_{64} = 16\,m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\, \xi^i\, \xi^j\, \xi^2 \left(\frac{\mathcal{I}^R}{m}\,\frac{\mathcal{I}^V}{m}\right) \varphi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\mathcal{I}^R\, d\mathcal{I}^v\, d\vec{\xi}.$$

The first of them has the form $B^{ij}_{14} = b_{14}\, \delta^{ij}$ whose trace is

$$3\, b_{14} = 4\,\pi\, m\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2}\, s^{10}\, d\,s\right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}}\, \varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}}\, \psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) = 9!!\, \rho \left(\frac{p}{\rho}\right)^4,$$

where eq. (24) has been used.
The second of them has the form $B^{ij}_{24} = b_{24}\, \delta^{ij}$ whose trace is

$$3\, b_{24} = 4\,\pi\, m\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2}\, s^{8}\, d\,s\right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}}\, \mathcal{I}^R\, \varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right) \frac{6}{m} \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}}\, \psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) = 7!!\, \rho \left(\frac{p}{\rho}\right)^4 \frac{6}{k_B\,T}\,\frac{B_R}{A_R},$$

where eq. (24) has been used.
The third of them is similar to the second, with $\mathcal{I}^V$ instead of $\mathcal{I}^R$; so it has the form $B^{ij}_{34} = b_{34}\, \delta^{ij}$ with

$$3\, b_{34} = 7!!\, \rho \left(\frac{p}{\rho}\right)^4 \frac{6}{k_B\,T}\,\frac{B_V}{A_V}.$$

The fourth of them has the form $B^{ij}_{44} = b_{44}\, \delta^{ij}$ whose trace is

$$3\, b_{44} = 4\,\pi\, m\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2}\, s^{6}\, d\,s\right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}} \left(\mathcal{I}^R\right)^2 \varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right) \frac{8}{m^2} \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}}\, \psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) = 5!!\, \rho \left(\frac{p}{\rho}\right)^4 \frac{8}{(k_B\,T)^2}\,\frac{C_R}{A_R},$$

where eq. (24) has been used.
The fifth of them is similar to the fourth, with $\mathcal{I}^V$ instead of $\mathcal{I}^R$; so it has the form $B^{ij}_{54} = b_{54}\, \delta^{ij}$ with

$$3\, b_{54} = 5!!\, \rho \left(\frac{p}{\rho}\right)^4 \frac{8}{(k_B\,T)^2}\,\frac{C_V}{A_V}.$$

The last of them has the form $B^{ij}_{64} = b_{64}\, \delta^{ij}$ whose trace is

$$3\, b_{64} = 4\,\pi\, m\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2}\, s^{6}\, d\,s\right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}}\, \mathcal{I}^R\, \varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right) \frac{16}{m^2} \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}}\, \mathcal{I}^V\, \psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) = 5!!\, \rho \left(\frac{p}{\rho}\right)^4 \frac{16}{(k_B\,T)^2}\,\frac{B_R}{A_R}\,\frac{B_V}{A_V},$$

where eq. (24) has been used.
So we have found that $\psi_5 = b_{14} + b_{24} + b_{34} + b_{44} + b_{54} + b_{64}$, i.e.,

$$\psi_5 = 5\,\rho\left(\frac{p}{\rho}\right)^4\left[63 + \frac{42}{k_B\,T}\,\frac{B_R}{A_R} + \frac{42}{k_B\,T}\,\frac{B_V}{A_V} + \frac{8}{(k_B\,T)^2}\,\frac{C_R}{A_R} + \frac{8}{(k_B\,T)^2}\,\frac{C_V}{A_V} + \frac{16}{(k_B\,T)^2}\,\frac{B_R}{A_R}\,\frac{B_V}{A_V}\right],$$

which gives the above reported expression.

- Let us consider now $B_1 = A_{2IE} + B_{11*} + B_{21*} + B_{31*}$ with

$$B_{11*} = \frac{4}{m}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{\chi_E}{k_B}}\left(\mathcal{I}^R\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi} = = \frac{4}{m}4\,\pi\,e^{-1-\frac{m}{k_B}\lambda_{IE}}\left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2}\,s^2\,d\,s\right)\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}}\left(\mathcal{I}^R\right)^2\varphi(\mathcal{I})\,d\mathcal{I}\right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}}\psi(\mathcal{I})\,d\mathcal{I}\right) = \frac{4}{m}\pi\,e^{-1-\frac{m}{k_B}\lambda_{IE}}\left(\frac{2\,k_B\,T}{m}\right)^{\frac{3}{2}}\sqrt{\pi}\,C_R\,A_V = 4\,\rho\left(\frac{k_B\,T}{m}\right)^2\gamma_R\,,$$

$$B_{21*} = \frac{4}{m}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{\chi_E}{k_B}}\left(\mathcal{I}^V\right)^2\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi},$$

which is similar to the previous one, with $\mathcal{I}^V$ instead of $\mathcal{I}^R$; so it is

$$B_{21*} = 4\,\rho\left(\frac{k_B\,T}{m}\right)^2\gamma_V\,;$$

$$B_{31*} = \frac{8}{m}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{\chi_E}{k_B}}\,\mathcal{I}^R\mathcal{I}^V\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi} = = \frac{8}{m}4\,\pi\,e^{-1-\frac{m}{k_B}\lambda_{IE}}\left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2}\,s^2\,d\,s\right)\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}}\mathcal{I}^R\varphi(\mathcal{I})\,d\mathcal{I}\right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}}\mathcal{I}^V\psi(\mathcal{I}^V)\,d\mathcal{I}^V\right) = \frac{8}{m}\pi\,e^{-1-\frac{m}{k_B}\lambda_{IE}}\left(\frac{2\,k_B\,T}{m}\right)^{\frac{3}{2}}\sqrt{\pi}\,B_R\,B_V = 8\,\rho\left(\frac{k_B\,T}{m}\right)^2\beta_R\,\beta_V\,,$$

where eq. (24) has been used. So we have found the above reported expression.

- Let us consider now $B_2 = B_{12} + B_{22} + B_{32} + B_{42} + B_{52} + B_{62}$ with

$$B_{12} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^6\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi},$$

$$B_{22} = 6\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E}\,\xi^4\,\mathcal{I}^R\,\varphi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\mathcal{I}^R\,d\mathcal{I}^v\,d\vec{\xi},$$

$$B_{32} = 6 \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \xi^4 \mathcal{I}^V \varphi(\mathcal{I}^R) \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{42} = \frac{8}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \xi^2 \left(\mathcal{I}^R\right)^2 \varphi(\mathcal{I}^R) \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{52} = \frac{8}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \xi^2 \left(\mathcal{I}^V\right)^2 \varphi(\mathcal{I}^R) \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{62} = \frac{16}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \xi^2 \mathcal{I}^R \mathcal{I}^R \varphi(\mathcal{I}^R) \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi}.$$

Now we have

$$B_{12} = 4\,\pi\, m\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2k_B T} s^2} s^8 \, d\,s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 7!!\, \rho \left(\frac{p}{\rho}\right)^3,$$

where eq. (24) has been used.

$$B_{22} = 24\,\pi\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2k_B T} s^2} s^6 \, d\,s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \mathcal{I}^R \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 6\,(5!!)\, \rho \left(\frac{p}{\rho}\right)^3 \beta_R .$$

B_{32} is similar to the previous one, with $\mathcal{I}^V$ instead of $\mathcal{I}^R$; so it is

$$B_{32} = 6\,(5!!)\, \rho \left(\frac{p}{\rho}\right)^3 \beta_V .$$

$$B_{42} = \frac{32}{m} \pi\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2k_B T} s^2} s^4 \, d\,s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \left(\mathcal{I}^R\right)^2 \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 24\, \rho \left(\frac{p}{\rho}\right)^3 \gamma_R .$$

B_{52} is similar to the previous one, with $\mathcal{I}^V$ instead of $\mathcal{I}^R$; so it is

$$B_{52} = 24\, \rho \left(\frac{p}{\rho}\right)^3 \gamma_V .$$

$$B_{62} = \frac{64}{m} \pi\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2k_B T} s^2} s^4 \, d\,s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B T}} \mathcal{I}^R \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B T}} \mathcal{I}^V \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 48\, \rho \left(\frac{p}{\rho}\right)^3 \beta_R \beta_V .$$

So we have found the above reported expression for B_2.

- Let us consider now $B_3 = B_{13} + B_{23} + B_{33} + B_{43} + B_{53} + B_{63}$ with

$$B_{13} = m \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \xi^8 \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{23} = 8 \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \xi^6 \, \mathcal{I}^R \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{33} = 8 \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \xi^6 \, \mathcal{I}^V \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{43} = \frac{16}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \xi^4 \, \left(\mathcal{I}^R\right)^2 \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{53} = \frac{16}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \xi^4 \, \left(\mathcal{I}^V\right)^2 \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi},$$

$$B_{63} = \frac{32}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-1-\frac{m}{k_B}\chi_E} \, \xi^4 \, \mathcal{I}^R \, \mathcal{I}^V \, \varphi(\mathcal{I}^R) \, \psi(\mathcal{I}^V) \, d\mathcal{I}^R \, d\mathcal{I}^v \, d\vec{\xi}.$$

We have

$$B_{13} = 4\,\pi\, m\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2} \, s^{10} \, d\,s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}} \, \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}} \, \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 9!!\, \rho \left(\frac{p}{\rho}\right)^4 .$$

$$B_{23} = 32\,\pi\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2} \, s^8 \, d\,s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}} \, \mathcal{I}^R \, \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}} \, \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 8\,(7!!)\, \rho \left(\frac{p}{\rho}\right)^4 \beta_R .$$

B_{33} is similar to the previous one, with $\mathcal{I}^V$ instead of $\mathcal{I}^R$; so it is

$$B_{33} = 8\,(7!!)\, \rho \left(\frac{p}{\rho}\right)^4 \beta_V .$$

$$B_{43} = \frac{64}{m}\,\pi\, e^{-1-\frac{m}{k_B}\lambda_{IE}} \left(\int_0^{+\infty} e^{-\frac{m}{2\,k_B\,T}s^2} \, s^6 \, d\,s \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_B\,T}} \, \left(\mathcal{I}^R\right)^2 \, \varphi(\mathcal{I}^R) \, d\mathcal{I}^R \right) \left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_B\,T}} \, \psi(\mathcal{I}^V) \, d\mathcal{I}^V \right) = 16\,(5!!)\, \rho \left(\frac{p}{\rho}\right)^4 \gamma_R .$$

B_{53} is similar to the previous one, with $\mathcal{I}^V$ instead of $\mathcal{I}^R$; so it is

$$B_{53} = 16\,(5!!)\,\rho\left(\frac{p}{\rho}\right)^4 \gamma_V\,.$$

$$B_{63} = \frac{128}{m}\,\pi\, e^{-1-\frac{m}{k_B}\lambda_{IE}}\left(\int_0^{+\infty} e^{-\frac{m}{2k_BT}s^2} s^6\, d\,s\right)\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^R}{k_BT}}\mathcal{I}^R\varphi(\mathcal{I}^R)\, d\mathcal{I}^R\right)$$
$$\left(\int_0^{+\infty} e^{-\frac{\mathcal{I}^V}{k_BT}}\mathcal{I}^V\psi(\mathcal{I}^V)\, d\mathcal{I}^V\right) = 32\,(5!!)\,\rho\left(\frac{p}{\rho}\right)^4\beta_R\,\beta_V\,.$$

So we have found the above reported expression for B_3.

- Let us consider now B_{2V}.

It is similar to A_{2IE} but there is the further factor $\frac{2\mathcal{I}^V}{m}$. So we obtain

$$B_{2V} = 8\pi\, e^{-1-\frac{m}{k_B}\lambda_I}\left[\frac{1}{2}\left(\frac{2\,k_BT}{m}\right)^{\frac{7}{2}}\frac{15}{8}\sqrt{\pi}\,A_R(T)B_V(T)+\right.$$
$$\left.+\frac{4}{m}\frac{1}{2}\left(\frac{2\,k_BT}{m}\right)^{\frac{5}{2}}\frac{3}{4}\sqrt{\pi}\,(B_RB_V + C_VA_R)\right]\,,$$

which, divided by eq. (24) gives the above reported expression for B_{2V}.

- Let us consider now B_{3V}.

It is similar to A_{0E} but there is the further factor $\left(\frac{2\mathcal{I}^V}{m}\right)^2$. So we obtain

$$B_{3V} = \frac{4}{m}\pi\, e^{-1-\frac{m}{k_B}\lambda_I}\left(\frac{2\,k_BT}{m}\right)^{\frac{3}{2}}\sqrt{\pi}\,A_R(T)C_V(T) = \frac{4\,\rho}{m^2}\frac{C_V}{A_V} = 4\,\rho\left(\frac{p}{\rho}\right)^2\gamma_V\,,$$

which is the above reported expression for B_{3V}.

- Let us conclude with $B^{ij}_{2V} = \psi_6\,\delta^{ij}$.

It is similar to A^{ij}_{1IE} but there is the further factor $\frac{2\mathcal{I}^V}{m}$. So we have

$$3\frac{\psi_6}{\rho} = \left(\frac{k_BT}{m}\right)^2\left[15\,A_R\,B_V + \frac{6}{k_BT}\,(B_V\,B_R + A_R\,C_V)\right]\frac{1}{A_R\,A_V}\,\frac{2}{m}\,,$$

i.e.,

$$\psi_6 = 2\,\rho\left(\frac{p}{\rho}\right)^3\left(5\beta_V + 2\,\beta_V\,\beta_R + 2\,\gamma_V\right)\,,$$

from which the above reported expression of B^{ij}_{2V} follows.

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