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# An Asymptotic Approach of The Crack Extension In Linear Piezoelectricity

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**Abstract:** As a result of a theoretical technique for elucidating the fracture mechanics of piezoelectric materials, this paper provides, on the basis of the three-dimensional model of thin plates, an asymptotic behavior in the Griffith's criterion for a weakly anisotropic thin plate with symmetry of order six, through a mathematical analysis of perturbations due to the presence of a crack. It is particularly established, in this work, the effects of both electric field and singularity of the in-plane mechanical displacement on the piezoelectric energy.
**Keywords:** Griffith's criterion; J-integral; Piezoelectricity; Asymptotic expansion; Crack extension; Mathematical analysis

## I. INTRODUCTION

In accordance with the conclusion of a one of our papers (Nianga, 2006), a rigorous mathematical approach of the J-piezoelectric integral (Rice, 1968,a,b) on the basis of Gol'denveizer's asymptotic integration methods is formulated. This paper arises from a number of investigations into the modelling of Cracks and piezoelectric behaviors (Bui, 1974, 1978; Parton, 1976; Chen and Lu, 2003; Dascalu and Maugin, 1994, 1995a,b; Destuynder, 1982, 1986; Nianga, 1996; Destyunder and Djaoua, 1981; Attou and Maugin, 1987; Maugin and Attou, 1990). Under these analyses, the three-dimensional models of piezoelectric plates are formulated as formal power series expansions. More generally, this paper leans on the extension of the Destuynder's model of elastic plates to piezoelectric materials, due to Maugin and Attou (Attou and Maugin, 1987; Maugin and Attou, 1990), without any a priori assumptions on the form of the unknowns. This paper is structured as follows: the basic three dimensional equations of piezoelectric thin plates are introduced in Section 2 and the perturbation of the plate under the Griffith's criterion is given in Section 3. Section 4 is devoted to the asymptotic behavior of the three-dimensional solution of the plate, and in Section 5, is presented an analogous study applied to the Griffith's criterion. The conclusion is given in Section 6.

## II. BASIC EQUATIONS OF LINEAR PIEZOELECTRIC THIN PLATE

Let us consider a thin three-dimensional piezoelectric plate with a sufficiently smooth boundary $\Gamma^{\varepsilon}$, occupying the open subset $\Omega^{\varepsilon}$ of $\mathbf{R}^3$ in its reference configuration. Let be $\gamma = \partial\omega$, the boundary of its median plane $\omega$. The thickness of this plate is $2\varepsilon$, where $\varepsilon$ verifies $(\varepsilon << 1)$, and has been already taken as parameter in the asymptotic expansion of electroelastic fields [ ]. $\Gamma_+^{\varepsilon} = \omega \times \{+\varepsilon\}$ and $\Gamma_-^{\varepsilon} = \omega \times \{-\varepsilon\}$ are the upper and lower faces of the plate, respectively; and $\Gamma_L^{\varepsilon} = \gamma \times ]-\varepsilon, +\varepsilon[$ is its lateral contour, with $\Gamma_{\varepsilon} = \Gamma_L^{\varepsilon} \cup \Gamma_+^{\varepsilon} \cup \Gamma_-^{\varepsilon}$ and $\Omega^{\varepsilon} = \omega \times ]-\varepsilon, +\varepsilon[$.

### 2.1. Local equations

The piezoelectric theory derives from a coupling between Maxwell's equations of electromagnetism and elastic stress equations of motion. In the quasi-electrostatic approximation, when body forces are neglected and free charges are absent, the equations of static equilibrium and Maxwellian equations are, respectively, of the form:

$$\nabla \cdot \sigma = 0 \qquad \text{in } \Omega^{\varepsilon} \tag{1}$$

$$\begin{cases} \nabla \cdot D = 0 \\ \nabla \times E = 0 \end{cases} \qquad \text{in } \Omega^{\varepsilon} \tag{2}$$

Where $\sigma$ is the stress tensor, D is the electric displacement vector, and E is the electric field vector. Therefore, it can be written:

$$E = -\nabla\varphi \qquad \text{in } \Omega^{\varepsilon} \tag{3}$$

Where $\varphi$ denotes an electric scalar potential. Generally, in the plate problem, the following boundary conditions are assumed:

$$u_i = 0 \tag{5}$$

$$\sigma_{ij} n_j^{\pm} = T_i^{\pm} \tag{6}$$

$$n \cdot [\![D]\!]_{\Gamma_L^{\varepsilon}} = q_0 \,;\; n^{\pm} \cdot [\![D]\!]_{\Gamma_{\pm}^{\varepsilon}} = w_{\pm} \tag{7}$$

Where $u_i$ (i=1, 2, 3) are the components of the elastic displacement vector, and where $n = \{n_1, n_2, n_3\}$ denotes the outward normal to $\Gamma^{\varepsilon}$, with $[\![\bullet]\!]$ expressing the jump of □at the considered face or contour. The plate is clamped on its lateral contour: condition (5); traction forces $T^{\pm}$ are imposed on the faces $\Gamma_{\pm}^{\varepsilon}$: condition (6); and electric charges may accumulate on the boundary $\Gamma^{\varepsilon}$ that is $w_{\pm}$ on the upper and lower faces and, $q_0$ on the lateral contour: condition (7). To express the continuity of the tangential component of the electric field E through $\Gamma^{\varepsilon}$, the following condition may be written:

$$[\![\varphi]\!]_{\Gamma_{\varepsilon}} = 0 \tag{8}$$

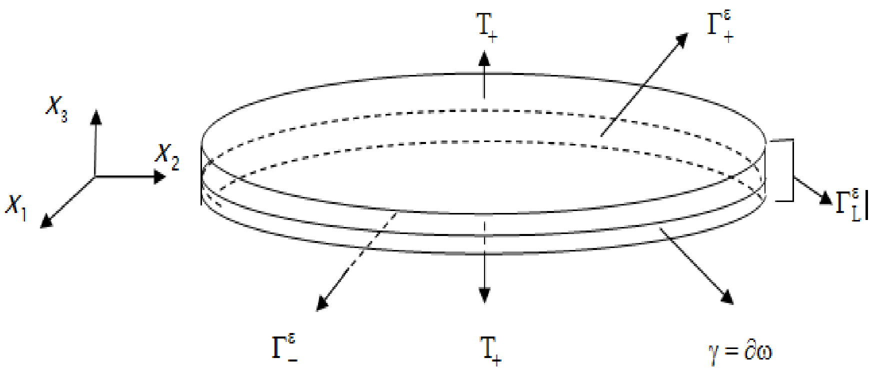


**Fig.1. A thin piezoelectric plate**

In the particular case of a piezoelectric thin plate of hexagonal 6mm symmetry, the constitutive equations in which the electroelastic coupling takes place, are written as follows (see Parton, 1976; Attou and Maugin, 1987; Maugin and Attou, 1990):

$$s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = a_{ijkl}\sigma_{kl} + b_{jip}D_p \tag{9}$$

$$\phi_{,k} = \tilde{a}_{klp}\sigma_{lp} + \tilde{b}_{kj}D_j \tag{10}$$

Where $a_{ijkl}, \tilde{a}_{klp}, b_{jip}$ and $\tilde{b}_{kj}$ are material coefficients assumed to be independent of the thickness $2\varepsilon$ of the plate, (i, j, k, p = 1, 2, 3); and where $s_{ij}$ are the strain tensor components.

### 2.2. Global equations – Variational formulation

Let $v, \tau, \delta$ and $\psi$ be test-functions of same nature as $u, \sigma, D$ and $\varphi$ respectively; the electromechanical generalization of Hellinger-Reissner principle may be expressed through the following problem:

**Problem** $P^\varepsilon$. Find $(\sigma, D; u, \varphi)$ in the space $(\Sigma^\varepsilon \times \Delta^\varepsilon) \times (V^\varepsilon \times \Phi^\varepsilon)$ such that:

$$\begin{cases} \forall(\tau,\delta) \in (\Sigma^\varepsilon \times \Delta^\varepsilon),\ A^\varepsilon(\sigma, D; \tau, \delta) + B^\varepsilon(\tau, \delta; u, \varphi) = 0 & (11) \\ \forall(v,\psi) \in (V^\varepsilon \times \Phi^\varepsilon), \qquad B^\varepsilon(\sigma, D; v, \psi) = F^\varepsilon(v, \psi) & (12) \end{cases}$$

Where

$$A^\varepsilon = \int_{\Omega^\varepsilon} [(a_{ijkl}\sigma_{ij}\tau_{kl} + b_{ijp}D_p\tau_{ij}) + (\tilde{a}_{klp}\sigma_{lp}\delta_k + \tilde{b}_{kj}D_j\delta_k)d\Omega \tag{13}$$

$$B^\varepsilon = -\int_{\Omega^\varepsilon} (\tau_{ij}u_{j,i}\tau_{kl} + \varphi_{,k}D_p\delta_k)d\Omega \tag{14}$$

$$F^\varepsilon = -(\int_{\Gamma^\varepsilon_\pm} (w_\pm\psi\, ds + \int_{\Gamma^\varepsilon_L} q_0\psi\, ds + \int_{\Gamma^\varepsilon_\pm} T_\pm \cdot v\, ds) \tag{15}$$

And

$$V^\varepsilon = \{v; v = \{v_i\}, i = 1,2,3; v_i \in H^1(\Omega^\varepsilon), v|_{\Gamma^\varepsilon_L} = 0\} \tag{16}$$

$$\Sigma^\varepsilon = \{\tau; \tau = \{\tau_{ij} = \tau_{ji}\}, i = 1,2,3; \tau_{ij} \in L^2(\Omega^\varepsilon)\} \tag{17}$$

$$\Delta^\varepsilon = \{\delta; \nabla \cdot \delta \in L^2(\Omega^\varepsilon)\} \tag{18}$$

$$\Phi^\varepsilon = \{\psi; \psi \in H^1(\Omega^\varepsilon)\} \tag{19}$$

Therefore, the piezoelectric energy may be expressed as:

$$W = \frac{1}{2}\int_{\Omega^\varepsilon} \left(\sigma_{ij}u_{j,i} + D_i\phi_{'i}\right) d\Omega = -\frac{1}{2}B^\varepsilon(\sigma, D; u, \varphi) \tag{20}$$

Where the quantity defined by

$$w(u,\varphi) = \frac{1}{2}\sigma_{ij}u_{j,i} + D_i\varphi_{,i} \tag{21}$$

Represents the electric enthalpy of the system; $(\sigma, D; u, \varphi)$ being the solution of the piezoelectric plate variational problem $P^\varepsilon$.

## III. FRACTURE MECHANICS AND PERTURBATIONS OF THE OPEN SUBSET $\Omega^\varepsilon$

Let us now consider that the plate contains a straight crack (see Fig.2).

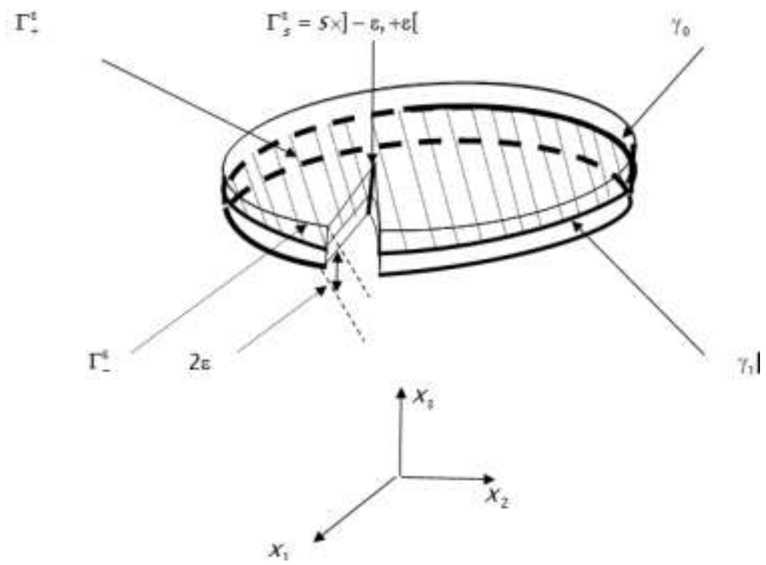


**Fig.2.** A thin three-dimensional piezoelectric plate with crack

$\Gamma_s^\varepsilon$ is the line of the crack-front ; $\gamma_0$ and $\gamma_1$ are the two parts of the boundary $\gamma$ of the median plane $\omega$, such that $\gamma = \gamma_0 + \gamma_1$. Let us now suppose a virtual kinematics $\theta = (\theta_i)_{i=1,2,3}$ for the crack tip, that is, an elastic displacement field of geometrical points characterizing the latter, and verifying the following conditions:

(a) $\theta = (\theta_1, 0, 0)$ (22)

The sense of the field $\theta$ is then the positive direction of the $x_1$ - axis.

(b) $\theta_1 = \theta_1(x_1, x_2) \in W^{2,\infty}(\varpi)$ (23)

With

$\theta_1 = 1 \text{ in } V_i(s)$ (24)

(c) $\text{Supp}(\theta_1) \cap \Gamma_0^\varepsilon = \varnothing$ (25)

$\text{Supp}(\theta_1) \cap \text{Supp}(T^\pm) = \varnothing$ (26)

$\text{Supp}(\theta_1) \subset V_i(\Gamma_S^\varepsilon)$ (27)

Where $V_i(s)$ is a vicinity of $s$ (see Fig.2); $V_i(\Gamma_S^\varepsilon)$ that of $\Gamma_S^\varepsilon$, and where $\text{Supp}(f)$ is a set such that: $f \equiv 0 \text{ in } \text{Supp}(f)$, for any scalar function $f$. The perturbations of the open subset $\Omega^\varepsilon$ are then defined by introducing, for each real parameter $\eta > 0$, the following function [Destuynder, 1986; Destuynder and Djaoua, 1981; Destuynder, 1982; Attou and Maugin, 1987; Maugin and Attou, 1990]

$F^\eta = I + \eta\theta$ (28)

Where I is the identity of $\mathbf{R}^3$.Then, $F^\eta$ transforms the open subset $\Omega^\varepsilon$ into an open $\Omega_\eta^\varepsilon$ in which, the associated piezoelectric energy is defined as follows:

$W\left(\Omega_\eta^\varepsilon\right) = \frac{1}{2}\int_{\partial\Omega_\eta^\varepsilon} \sigma_{ij}^\eta u_j n_i - \frac{1}{2}\int_{\partial\Omega_\eta^\varepsilon} D_k^\eta \phi^\eta n_k$ (29)

$(\sigma^\eta, D^\eta; u^\eta, \varphi^\eta)$ represents the solution of Problem $P_\eta^\varepsilon$ analogous, in the open subset $\Omega_\eta^\varepsilon$, to Problem $P^\varepsilon$ defined on $\Omega^\varepsilon$; with:

$$(\sigma^\eta, D^\eta; u^\eta, \varphi^\eta)\Big|_{\eta=0} = (\sigma, D; u, \varphi) \tag{30}$$

And

$$\begin{aligned}\frac{dW\left(\Omega_\eta^\varepsilon\right)}{d\eta}(\theta)\Bigg|_{\eta=0} &= \lim_{\eta\to 0}\frac{W\left(\Omega_\eta^\varepsilon\right)-W\left(\Omega^\varepsilon\right)}{\eta} \\ &= -\frac{1}{2}\int_{\Omega^\varepsilon}\sigma_{ij}\partial_i u_j \partial_\ell \theta_\ell d\Omega + \int_{\Omega^\varepsilon}\sigma_{il}\partial_k u_l \partial_i \theta_k d\Omega \\ &\quad -\frac{1}{2}\int_{\Omega^\varepsilon} D_i \varphi_{,i}\partial_\ell \theta_\ell d\Omega \quad + \quad \int_{\Omega^\varepsilon} D_i \varphi_{,j}\partial_i \theta_j d\Omega\end{aligned} \tag{31}$$

Where $\dfrac{dW\left(\Omega_\eta^\varepsilon\right)}{d\eta}(\theta)$ represents the energy required to fracture the material in the direction $\theta$. According to Griffith's criterion (Bui, 1974, 1978; Rice, 1968a,b), the energy that allows the crack with unit area to grow, is a material constant $\gamma_C$. Due to the fact that the transition from $\Omega^\varepsilon$ to $\Omega_\eta^\varepsilon\big|_{\eta=1}$ produces an unit area, the crack will spread if only if:

$$\frac{dW\left(\Omega_\eta^\varepsilon\right)}{d\eta}(\theta) \le -2\gamma_C < 0 \tag{32}$$

With

$$\int_{\Gamma_S^\varepsilon}\theta_1 ds = \text{mes}\left(\Gamma_S^\varepsilon\right)\theta_1 = 1 \tag{33}$$

Therefore, $\theta_1$ varies as the inverse of the thickness $2\varepsilon$ of the plate. So, the condition of the crack propagation only depends, at this stage of our study, on the knowledge of the value $\theta_1$ evaluated on the line of the crack front $\Gamma_S^\varepsilon$.

**Remark.1.** In fact, (31) only depends on singularities of the three-dimensional solution of the problem of the plate when the crack tip is approached, and doesn't depend on $\theta$. Nevertheless, as the question of these three-dimensional singularities is not elucidated yet, we then propose an asymptotic analysis of Griffith's criterion according to the thickness $2\varepsilon$ of the plate. However, we will first of all, analyze the asymptotic behavior of the three-dimensional solution for a thin piezoelectric plate.

## IV. ASYMPTOTIC BEHAVIOR OF THE THREE-DIMENSIONAL SOLUTION

Let us introduce an open subset $\Omega = \omega \times \left]-1,+1\right[$ of $\mathbf{R}^3$, with boundaries:

$$\Gamma_\pm = \omega \times \{\pm 1\} \tag{34}$$

$$\Gamma_1 = \gamma_1 \times \left]-1,+1\right[ \tag{35}$$

$$\Gamma_2 = \gamma_2 \times \left]-1,+1\right[ \tag{36}$$

Where $\gamma_1$ and $\gamma_2$ denote the two complementary parts of the boundary $\partial\omega = \gamma$ of the median plane, such that $\gamma_1 \cup \gamma_2 = \gamma$ . If we define, for each $\varepsilon$ , the following bijection:

$$\pi^{\varepsilon} : x = (x_1, x_2, x_3) \in \overline{\Omega} \longrightarrow \pi^{\varepsilon}(x) = x^{\varepsilon} = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^{\varepsilon} \tag{37}$$

Then, for any function f defined on $\overline{\Omega}^{\varepsilon}$ , we associate the function $f^{\varepsilon}$ defined on $\overline{\Omega}$ by:

$$f^{\varepsilon}(x) = f \circ \pi^{\varepsilon}(x) \tag{38}$$

Furthermore, the unknown fields $\sigma, D, u$ and $\varphi$ defined on $\Omega^{\varepsilon}$ and verifying Problem $P^{\varepsilon}$ of the three-dimensional piezoelectric plate can be transformed respectively into the fields $\sigma^{\varepsilon}, D^{\varepsilon}, u^{\varepsilon}$ and $\varphi^{\varepsilon}$, now defined on $\Omega$ , as follows:

$$\begin{cases} \sigma^{\varepsilon}_{\alpha\beta} = \sigma_{\alpha\beta} \circ \pi^{\varepsilon}, & u^{\varepsilon}_{\alpha} = u_{\alpha} \circ \pi^{\varepsilon}, & (T_3^{\pm})^{\varepsilon} = \varepsilon^{-1} T_3^{\pm} \circ \pi^{\varepsilon} \\ \sigma^{\varepsilon}_{33} = \varepsilon^{-2} \sigma_{33} \circ \pi^{3}, & u^{\varepsilon}_{3} = \varepsilon\, u_3 \circ \pi^{\varepsilon}, & D^{\varepsilon}_{\alpha} = D_{\alpha} \circ \pi^{\varepsilon} \\ \sigma^{\varepsilon}_{\alpha 3} = \varepsilon^{-1} \sigma_{\alpha 3} \circ \pi^{3}, & (T_{\alpha}^{\pm})^{\varepsilon} = \varepsilon^{-1}\, T_{\alpha}^{\pm} \circ \pi^{\varepsilon}, & D^{\varepsilon}_{3} = \varepsilon^{-1} D_3 \circ \pi^{\varepsilon} \\ W^{\varepsilon}_{\pm} = \varepsilon^{-1} W_{\pm} \circ \pi^{\varepsilon}, & \alpha, \beta = 1, 2. & \end{cases} \tag{39}$$

But, if we take the following asymptotic expansions into account:

$$\left(\sigma^{\varepsilon}, D^{\varepsilon}, u^{\varepsilon}, \phi^{\varepsilon}\right) = \left(\sigma^{(0)}, D^{(0)} ; u^{(0)}, \phi^{(0)}\right) + \varepsilon\left(\sigma^{(1)}, D^{(1)}, u^{(1)}, \phi^{(1)}\right) + \dots \tag{40}$$

The first term $\left(\sigma^{(0)}, D^{(0)} ; u^{(0)}, \phi^{(0)}\right)$ then verifies the following equations [Attou and Maugin, 1987; Maugin and Attou, 1990]:

$$A_0\left(\sigma^{(0)}, D^{(0)} ; u^{(0)}, \phi^{(0)}\right) + B_0\left(\tau, \delta ; u^{(0)}, \phi^{(0)}\right) = 0 \text{ for all } (\tau, \delta) \in (\Sigma \times \Delta) \tag{41}$$

$$B_0\left(\sigma^{(0)}, D^{(0)} ; v, \varphi\right) = F_0(v, \varphi) \text{ for all } (v, \varphi) \in (V \times \Phi) \tag{42}$$

Where $A_0, B_0$ and $F_0$ represent bilinear forms, with:

$$\Sigma = \Sigma^{\varepsilon}\Big|_{\varepsilon=1}, \Delta = \Delta^{\varepsilon}\Big|_{\varepsilon=1}, V = V^{\varepsilon}\Big|_{\varepsilon=1}, \Phi = \Phi^{\varepsilon}\Big|_{\varepsilon} \tag{43}$$

Exploiting directly Equations (41) and (42) from their local form in the median plane $\omega$, the asymptotic behavior of the three-dimensional solution of the problem for a bending piezoelectric thin plate can then be formulated through the following theorem:

**Theorem1.** When $\varepsilon$ tends to zero, the mechanics displacement $u_3^{\varepsilon}$ and the electric potential $\varphi^{\varepsilon}$ introduced in (39), strongly converge to $u_3^{0}$ and $\varphi^{(0)}$ in $H^1(\Omega)$ respectively; the latter being the unique solutions of the following equations:

$$\begin{cases} \frac{2}{3}\left[ a^{-1}_{1111}(u^{(0)}_{3,1111}+u^{(0)}_{3,2222})+(a^{-1}_{1111}+2a^{-1}_{1122})(u^{(0)}_{3,1112}+u^{(0)}_{3,1222}+2u^{(0)}_{3,1122}) \right] = 2\left\langle T_3 + T_{\alpha,\alpha} \right\rangle \\ u_3^{(0)} = u_3^{(0)}(x_1,x_2);\ u_3^{(0)} \in H_0^2(\omega) \end{cases} \tag{44}$$

$$u_3^{(0)} = \frac{\partial u_3^{(0)}}{\partial n} = 0 \text{ on } \gamma \tag{45}$$

$$\tilde{\varepsilon}_{11}\Delta_2\varphi^{(0)} = \langle w \rangle \text{ in } \omega \tag{46}$$

$$\tilde{\varepsilon}_{11}\frac{\partial \varphi^{(0)}}{\partial n} = 0 \text{ on } \gamma \tag{47}$$

Where $\tilde{\varepsilon}_{11}$ denotes a coefficient of electromechanical coupling, and where $\langle \cdot \rangle$ represents the mean value on the upper and lower faces of the plate. Furthermore, we get:

$$u^{\varepsilon}_{\alpha} \to u^{(0)}_{\alpha} = \tilde{u}^{(0)}_{\alpha} - x_3\partial_{\alpha}u^{(0)}_3 \text{ in the sense of } H^1(\Omega) \tag{48}$$

$$D^{\varepsilon}_{\alpha} \to D^{(0)}_{\alpha} = -\tilde{\varepsilon}_{11}\phi^{(0)}_{,\alpha} \text{ in the sense of } H^1(\Omega) \to \tag{49}$$

$$D^{\varepsilon}_3 \to D^{(0)}_3 = -w_- - \int_{-1}^{x_3} D^{(0)}_{\alpha,\alpha}(x_1,x_2,\xi)d\xi \text{ in the sense of } H^1(\Omega) \tag{50}$$

$$\varepsilon D^{\varepsilon}_3 \to 0 \text{ in the sense of } L^2(\Omega) \tag{51}$$

$$\sigma^{\varepsilon}_{\alpha\beta} \to \sigma^{(0)}_{\alpha\beta} = a^{-1}_{\alpha\beta\gamma\delta}u^{(0)}_{\gamma,\delta} \text{ in the sense of } L^2(\Omega) \tag{52}$$

$$\sigma^{\varepsilon}_{\alpha 3} \to \sigma^{(0)}_{\alpha 3} = -T^-_{\alpha} - \int_{-1}^{x_3} \sigma^{(0)}_{\alpha\beta,\beta}(x_1,x_2,\xi)d\xi \text{ in the sense of } L^2(]-1,+1[;H^{-1}(\omega)) \tag{53}$$

$$\varepsilon\sigma^{\varepsilon}_{\alpha 3} \to 0 \text{ in the sense of } L^2(\Omega) \tag{54}$$

$$\sigma^{\varepsilon}_{33} \to \sigma^{(0)}_{33} = -T^-_3 - \int_{-1}^{x_3} \sigma^{(0)}_{\alpha 3,\alpha}(x_1,x_2,\xi)d\xi \text{ in the sense of } L^2(]-1,+1[;H^{-2}(\omega)) \tag{55}$$

$$\varepsilon^2\sigma_{33} \to 0 \text{ in the sense of } L^2(\Omega) \tag{56}$$

$$\phi^{\varepsilon} \to \phi^{(0)} \text{ in the sense of } L^2\left(]-1,+1[\,; H^{-2}(\omega)\right) \tag{57}$$

## V. SYMPTOTIC BEHAVIOR IN THE GRIFFITH'S CRITERION

Taking the change of unknowns (38) and (39) into account, the energy required to fracture the plate in the direction $\theta$ becomes:

$$\begin{aligned} \frac{dW(\Omega^{\varepsilon})}{d\eta}(\theta) = & -\frac{1}{2}\int_{\Omega}\sigma^{\varepsilon}_{\alpha\beta}\partial_{\alpha}u^{\varepsilon}_{\beta}\partial_{\gamma}\tilde{\theta}_{\gamma}\, d\Omega + -\frac{1}{2}\int_{\Omega}\sigma^{\varepsilon}_{\alpha 3}\left(\partial_{\alpha}u^{\varepsilon}_3 + \partial_3 u^{\varepsilon}_{\alpha}\right)\partial_{\gamma}\tilde{\theta}_{\gamma}\, d\Omega \\ & -\frac{1}{2}\int_{\Omega}\sigma^{\varepsilon}_{33}\partial_3 u^{\varepsilon}_3\partial_{\gamma}\tilde{\theta}_{\gamma}\, d\Omega + \int_{\Omega}\sigma^{\varepsilon}_{\alpha\beta}\partial_{\gamma}u^{\varepsilon}_{\beta}\partial_{\alpha}\tilde{\theta}_{\gamma}\, d\Omega \\ & +\int_{\Omega}\sigma^{\varepsilon}_{\alpha 3}\partial_3 u^{\varepsilon}_{\beta}\partial_{\alpha}\tilde{\theta}_{\beta}\, d\Omega - \frac{1}{2}\int_{\Omega}D^{\varepsilon}_{\alpha}\varphi^{\varepsilon}_{,\alpha}\partial_{\gamma}\tilde{\theta}_{\gamma}\, d\Omega \\ & +\int_{\Omega}D^{\varepsilon}_{\alpha}\varphi^{\varepsilon}_{,\beta}\partial_{\alpha}\tilde{\theta}_{\beta}\, d\Omega + \int_{\Omega}D^{\varepsilon}_3\varphi^{\varepsilon}_{,\alpha}\partial_3\tilde{\theta}_{\alpha}\, d\Omega \end{aligned} \tag{58}$$

Where $\tilde{\theta} = \varepsilon\theta = (\varepsilon\theta_1, 0, 0) = (\tilde{\theta}_1, 0, 0)$. Furthermore, comparing (58) and Theorem 1, the asymptotic behavior in the Griffith's criterion then follows, through Theorem 2:

**Theorem2.** When $\varepsilon \to 0$, $\dfrac{dW\left(\Omega^{\varepsilon}\right)}{d\eta}(\theta)$ converges to

$$\begin{aligned}\frac{d\tilde{W}}{d\eta}\left(\tilde{\theta}\right) = &-\frac{1}{3}\int_{\omega} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta\alpha}u_3^{(0)}\partial_1\tilde{\theta}_1\, d\omega + \frac{2}{3}\int_{\omega} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta 1}u_3^{(0)}\partial_{\alpha}\tilde{\theta}_1\, d\omega \\ &+ \int_{\omega}\tilde{\varepsilon}_{11}\left\|\nabla_2\varphi^{(0)}\right\|^2_{L^2(\omega)}\partial_1\tilde{\theta}_1\, d\omega \quad + 2\int_{\omega}\tilde{\varepsilon}_{11}\ \varphi^{(0)}_{,\alpha}\varphi^{(0)}_{,1}\partial_{\alpha}\tilde{\theta}_1\, d\omega \\ &+ \frac{2}{3}\int_{\omega} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta}(\partial_1 u_3^{(0)}\partial_{\alpha}\tilde{\theta}_1)\, d\omega\end{aligned} \tag{59}$$

Where $\nabla_2$ is the in-plane gradient operator.

However $\dfrac{d\tilde{W}}{d\eta}\left(\tilde{\theta}\right)$ doesn't depend on the choice of the field $\theta$, as it will be shown in the following theorem and subsequently.

**Theorem3.** Let C be a region with the boundary ¶ C is a regular curve of the open $\omega$, including the tip S of the crack, and enclosed in a region where the field θ is constant. $\bar{C}$ denotes the complementary part of C in $\omega$, with boundary $\partial\bar{C}$; and **n** is the unit outward normal to $\bar{C}$. We have, for a weakly anisotropic piezoelectric plate:

$$\begin{aligned}\frac{d\tilde{W}}{d\eta}\left(\tilde{\theta}\right) = &-\frac{1}{3}\int_{\partial\bar{C}} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta\alpha}u_3^{(0)}\tilde{\theta}_1 n_1 + \frac{2}{3}\int_{\partial\bar{C}} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta 1}u_3^{(0)}\tilde{\theta}_1 n_{\alpha} \\ &+ \int_{\partial\bar{C}}\tilde{\varepsilon}_{11}\left\|\nabla_2\varphi^{(0)}\right\|^2_{L^2(\omega)}\tilde{\theta}_1 n_1 \quad + 2\int_{\partial\bar{C}}\tilde{\varepsilon}_{11}\ \varphi^{(0)}_{,\alpha}\varphi^{(0)}_{,1}\tilde{\theta}_1 n_{\alpha} d\omega \\ &+ \frac{2}{3}\int_{\partial\bar{C}} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_1 u_3^{(0)}\partial_{\alpha}\tilde{\theta}_1 n_{\beta}\end{aligned} \tag{60}$$

The proof is analogous to that of Theorem 5.3 (see P. Destuynder [ ]).

**Proof**. Since θ is constant in the open C, we then have

$$\frac{d\tilde{W}}{d\eta}\left(\tilde{\theta}\right) = 0 \text{ in C} \tag{61}$$

Furthermore, using Stokes' formula, Equation (59) can be written as follows:

$$\begin{aligned}\frac{d\tilde{W}}{d\eta}\left(\tilde{\theta}\right) &= -\frac{1}{3}\int_{\partial\bar{C}} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta\alpha}u_3^{(0)}\tilde{\theta}_1 n_1 + \frac{1}{3}\int_{\bar{C}}\partial_1(a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta\alpha}u_3^{(0)})\tilde{\theta}_1 \\ &+\frac{2}{3}\int_{\partial\bar{C}} a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta 1}u_3^{(0)}\tilde{\theta}_1 n_\alpha - \frac{2}{3}\int_{\bar{C}}\partial_\alpha(a^{-1}_{\alpha\beta\gamma\delta}\ \partial_{\gamma\delta}u_3^{(0)}\partial_{\beta 1}u_3^{(0)}\tilde{\theta}_1) \\ &+\frac{2}{3}\int_{\partial\bar{C}} a^{-1}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_3^{(0)}\partial_1 u_3^{(0)}\partial_\alpha\tilde{\theta}_1 n_\beta - \frac{2}{3}\int_{\partial\bar{C}} a^{-1}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_3^{(0)}\partial_1 u_3^{(0)}\tilde{\theta}_1 n_\alpha \\ &+\frac{2}{3}\int_{\bar{C}}\partial_\alpha(\partial_\beta a^{-1}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_3^{(0)}\partial_1 u_3^{(0)})\tilde{\theta}_1 + \int_{\partial\bar{C}}\tilde{\varepsilon}_{11}\left\|\nabla_2\varphi^{(0)}\right\|^2_{L^2(\omega)}\tilde{\theta}_1 n_1 \\ &+ 2\int_{\partial\bar{C}}\tilde{\varepsilon}_{11}\ \varphi^{(0)}_{,\alpha}\varphi^{(0)}_{,1}\tilde{\theta}_1 n_\alpha \qquad -2\int_{\bar{C}}\partial_\alpha(\tilde{\varepsilon}_{11}\ \varphi^{(0)}_{,\alpha}\varphi^{(0)}_{,1})\tilde{\theta}_1\end{aligned} \tag{62}$$

Since, for a weakly anisotropic piezoelectric plate, we have:

$$\partial_1(a^{-1}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_3^{(0)})\partial_{\alpha\beta}u_3^{(0)} = a^{-1}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_3^{(0)}\partial_{1\alpha\beta}u_3^{(0)} \tag{63}$$

It then follows, taking (44) into account:

$$\begin{aligned}&\frac{1}{3}\partial_1(a^{-1}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_3^{(0)}\partial_{\beta\alpha}u_3^{(0)}) - \frac{2}{3}\partial_\alpha(a^{-1}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_3^{(0)}\partial_{\beta 1}u_3^{(0)}) + \frac{2}{3}\partial_\alpha(\partial_\beta(a^{-1}_{\alpha\beta\gamma\delta})\partial_{\gamma\delta}u_3^{(0)}\partial_1 u_3^{(0)}) \\ &= \frac{2}{3}\partial_{\alpha\beta}(a^{-1}_{\alpha\beta\gamma\delta}u_3^{(0)})\partial_1 u_3^{(0)}\end{aligned} \tag{64}$$

But, as $\tilde{\theta}_1$ and $T_3$ or $T_\alpha$ are not identically non-zero in the same space, the part of $\frac{d\tilde{W}}{d\eta}\left(\tilde{\theta}\right)$ corresponding to $u_3^{(0)}$, then amounts only to terms expressed over $\partial\bar{C}$. Concerning the terms related to $\varphi^{(0)}$, it is sufficient to write:

$$\tilde{\varepsilon}_{11}\varphi^{(0)}_{,\alpha}\varphi^{(0)}_{,\beta}\partial_\alpha\tilde{\theta}_\beta = \tilde{\varepsilon}_{11}\varphi^{(0)}_{,1}\nabla_2\varphi^{(0)}\nabla_2\tilde{\tilde{\theta}} \tag{65}$$

Where $\tilde{\tilde{\theta}} = (\tilde{\theta}_1, \tilde{\theta}_1)$, in order to obtain the nullity of the integral over $\bar{C}$. Consequently, the proof of theorem3 then follows.

**Remark.2.** When the region C tends to the crack-tip, so that $\tilde{\theta}_1 = 1$, it is easy to establish that $\frac{d\tilde{W}}{d\eta}\left(\tilde{\theta}\right)$ only depends on the singularity of $u_3^{(0)}$ at this point, not on $\theta$.

## VI. CONCLUSION

It is established that the crack-extension criterion can be expressed only by the solution of the problem of linear piezoelectricity, in the median plane of the plate. Moreover, it is clear that the transverse shear, even in the Kirchoff-Love theory of plates, has a fundamental importance in the definition of the crack-extension criterion, as it has been established by Destuynder and Djaoua (Destyunder and Djaoua, 1981) in the context of linear elasticity. On the other hand, it is shown that the energy available to fracture, doesn't depend on the contour C centered on the crack-tip, but depends on the singularities of $u_3^{(0)}$ and $\phi^{(0)}$ at this point, and also on the effects of the field $E^{(0)} = -\nabla_2\varphi^{(0)}$ defined on the median plane of the plate.

## REFERENCES

[1]. Attou, D., Maugin, G.A., 1987. Une théorie asymptotique des plaques minces piézoélectriques. C.R. Acad. Sci. Paris, Série II 304 (15), 865-868.
[2]. Bui, H.D., 1974. Dual path independent integrals in the boundary-value problems of cracks. Eng. Fracture Mech. 6, p. 287-296.
[3]. Bui, H.D., 1978. Mécanique de la rupture fragile, Masson, Paris.
[4]. Chen, Y.H., Lu, T.J., 2003. Cracks and fracture in piezoelectrics. In: Aref, H., Van der Giessen (Eds.), Advances in Applied Mechanics, vol. 39. Academic Press, pp. 121-215.
[5]. Dascalu, C., Maugin, G.A., 1994. Energy-release rates and Path-independent Integrals on electroelastic crack propagation. Int. J. Eng. Sci. 32 (5) 775-765.
[6]. Dascalu, C., Maugin, G.A., 1995a. On the dynamic fracture of piezoelectric materials. Q. J. Mech. Appl. Math. 48 (Pt. 2) 237-255.
[7]. Dascalu, C., Maugin, G.A., 1995b. On the energy of electroelastic fracture. Z. Angew. Math. Phys. 46, 335-365 (1995).
[8]. Destuynder, Ph., 1982. Sur la propagation des fissures dans les plaques minces en flexion. J. Méca. Théo App. 1 (4), p 579 – 594.
[9]. Destuynder, Ph., 1986. Une théorie asymptotique des plaques minces en élasticité linéaire, Collection R.M.A.2. Masson, Paris.
[10]. Destuynder, Ph., Djaoua, M., 1981. Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile. Mat. Meth. Appl. Sci. 3, 70-87
[11]. Maugin, G.A., Attou, D., 1990. An asymptotic theory of thin piezoelectric plates. Q. J. Mech. Appl. Math. 43, (Pt. 3), 347 –362.
[12]. Nianga, J.M., 1996. Contribution à l'étude du comportement asymptotique des plaques minces en piézoélectricité linéaire. Université des Sciences et Technologiques de Lille.
[13]. Nianga, J.M., 2006. An extension of the J-integral to piezoelectric materials. Mech. Research. Communication. 33, 747-752.
[14]. Parton, V. Z., 1976. Fracture mechanics of piezoelectric materials. Acta Astronaut. 3, 671 – 683.
[15]. Rice, J.R., 1968a. A path independent integral and approximate analysis of strain concentrations by notches and cracks. J. Appl. Mech. 35, 375-386.
[16]. Rice, J.R., 1968b. In: Liebowitz, H. (Ed.), Fracture, Vol.3. Academic Press, p.191.
[17]. Suo, Z., Kuo, M., Barnett, D.M., Willis, J.R., 1992. Fracture mechanics for piezoelectric ceramics, J. Mech. Phys. Solids 40 (4), 739 – 765.