International Refereed Journal of Engineering and Science (IRJES)
ISSN (Online) 2319-183X, (Print) 2319-1821
Volume 6, Issue 3 (March 2017), PP.38- 44

Some Results of Compatible Mapping in Metric Spaces

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ABSTRACT: In this paper, we prove some common fixed point theorems of compatible mappings with the generalized contractive mappings in metric spaces and extents the results of J.K. Jang, J.K. Yun, N.J. Bae, J.H. Kim, D.M. Lee, S.M. Kang (Common Fixed Point Theorems of Compatible Mappings in Metric Spaces, International Journal of Pure and Applied Mathematics Vol 84, No. 1 (2013), 171-183) and some other authors.
Keywords: common fixed points, compatible mappings, complete metric spaces, gen- eralized contractive mappings.
2010 Mathematics Subject Classification: 47H10, 54H25.

I. Introduction

The most well-known fixed point theorem is so called Banachs fixed point theorem. For an extension of Banach's fixed point theorem, Hardy-Rogers [4], Rhoades [12] and many others introduced a more generalized contractive mappings.

In 1976, Jungck [5] initially proved a common fixed point theorem for commuting map- pings, which generalizes the well-known Banach's fixed point theorem. This result has been generalized, extended and improved by many authors (see [2], [3], [6]-[8], [10], [11], [13]-[16]) in various ways.

On the other hand, in 1982, Sessa [14] introduced a generalization of commutativity, which is called the weak commutativity, and proved some common fixed point theorems for weakly commuting mappings which generalize the results of Das-Naik [2].

In 1986, Jungck [6] introduced the concept of the more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. By employing compatible mappings instead of commuting mappings and using four mappings instead of three mappings, Jungck [7] extended the results of Khan-Imdad [10] and Singh-Singh [16].

Further, Cho-Yoo [1] and Kang-Kim [9] proved some fixed point theorems for compatible mappings.

In this paper, we prove some common fixed point theorems of compatible mappings with the generalized contractive mappings in metric spaces and also give some examples to illustrate our main theorems. These results generalize the results of Cho-Yoo [1], Jungck [7] and Kang-Kim [9].

II. Preliminaries

The following was introduced by Sessa [14].

Definition 2.1. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be weakly commuting mappings on X if $d(ABx, BAx) \le d(Ax, Bx)$ for all $x \in X$.

Clearly, commuting mappings are weakly commuting, but the converse is not necessarily true as in the following example:

Example 2.2. Let $X = [0, 1]$ with the Euclidean metric d. Define the mappings $A, B : X \to X$ by

$$Ax = \frac{1}{2}x, \quad Bx = \frac{x}{2+x}$$

for all $x \in X$, respectively.

The following was given by Jungck [6].

Definition 2.3. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be compatible mappings on X if

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some point $t \in X$.
Obviously, weakly commuting mappings are compatible, but the converse is not necessarily true as in the following example:

Example 2.4. Let $X = (-\infty, \infty)$ with the Euclidean metric d. Define the mappings A, B: $X \to X$ by

$$Ax = x^3, \quad Bx = 2 - x$$

for all $x \in X$, respectively.

We need the following lemmas for our main theorems, which were proved by Jungck [5] and [6].

Lemma 2.5. Let $\{y_n\}$ be a sequence in a metric space (X, d) satisfying

$$d(y_{n+1}, y_{n+2}) \le h d(y_n, y_{n+1})$$

for $n = 1, 2, \ldots$, where $0 < h < 1$. Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.6. Let A and B be compatible mappings from a metric space (X, d) into itself. Suppose that $At = Bt$ for some $t \in X$. Then $d(ABt, BAt) = 0$, that is, $ABt = BAt$.

Lemma 2.7. Let A and B be compatible mappings from a metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$. Then $\lim_{n\to\infty} BAx_n = At$ if A is continuous.

III. Fixed Point Theorems

Now, let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subset T(X), \quad B(X) \subset S(X) \qquad (3.1)$$

$$\begin{aligned} d(Ax, By) \le\ & p \max\left\{ d(Ax, Sx), d(By, Ty), \tfrac{1}{2}\left[d(Ax, Ty) + d(By, Sx)\right], d(Sx, Ty) \right\} \\ & + q \max\left\{ d(Ax, Sx), d(By, Ty) \right\} + r \max\left\{ d(Ax, Ty), d(By, Sx), \right. \\ & \left. \tfrac{1}{2}\left[d(Ax, By) + d(Ty, Sx)\right], d(Sx, Ty) \right\} \end{aligned} \qquad (3.2)$$

for all $x, y \in X$, where $0 < h = p + q + 2r < 1$ (p, q and r are non-negative real numbers). Then, for an arbitrary point x_0 in X, by (3.1), we choose a point x_1 in X such that $Tx_1 = Ax_0$ and, for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that, for

$$n = 0, 1, 2, \ldots,$$
$$y_{2n+1} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n} = Sx_{2n} = Bx_{2n-1} \qquad (3.3)$$

Lemma 3.1. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X.

Proof: Let $\{y_n\}$ be the sequence in X defined by (3.3). From (3.2), we have

$$
\begin{aligned}
d(y_{2n+1}, y_{2n+2}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\le p \max\Big\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \tfrac{1}{2}\big[d(y_{2n+1}, y_{2n+1}) \\
&\quad + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) \big] \Big\} \\
&\quad + q \max\big\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}) \big\} \\
&\quad + r \max\Big\{ d(y_{2n+1}, y_{2n+1}), d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), \\
&\quad \tfrac{1}{2}\big[d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n+1}) \big], d(y_{2n+1}, y_{2n+2}) \Big\} \qquad (3.4)
\end{aligned}
$$

where $0 < h = p + q + 2r < 1$. In (3.4), if $d(y_{2n+1}, y_{2n+2}) > d(y_{2n}, y_{2n+1})$ for some positive integer n, then we have

$$d(y_{2n+1}, y_{2n+2}) \le hd(y_{2n+1}, y_{2n+2}),$$

Which is a contradiction. Thus we have

$$d(y_{2n+1}, y_{2n+2}) \le hd(y_{2n}, y_{2n+1})$$

Similarly, we obtain

$$d(y_{2n}, y_{2n+1}) \le hd(y_{2n-1}, y_{2n})$$

It follows from the above facts that

$$d(y_{n+1}, y_{n+2}) \le hd(y_n, y_{n+1})$$

for $n = 1, 2, \ldots,$ where $0 < h < 1$. By Lemma 2.5, $\{y_n\}$ is a Cauchy sequence in X.
Now, we are ready to give our main theorems.

Theorem 3.2. Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (3.1) and (3.2). Suppose that
(3.5) one of A, B, S and T is continuous,
(3.6) the pairs A, S and B, T are compatible on X.
Then A, B, S and T have a unique common fixed point in X.

Proof: Let $\{y_n\}$ be the sequence in X defined by (3.3). By Lemma 3.1, $\{y_n\}$ is a Cauchy sequence and hence it converges to some point $z \in X$. Consequently, the subsequences $\{Ax_{2n}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to the point z.
Now, suppose that S is continuous. Since A and S are compatible on X, Lemma 2.7 gives that

$$S^2x_{2n} \to Sz, \ ASx_{2n} \to Sz \text{ as } n \to \infty$$

By (3.2), we obtain

$$
\begin{aligned}
d(ASx_{2n}, Bx_{2n-1}) &\le p \max\Big\{ d(ASx_{2n}, S^2x_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), \tfrac{1}{2}\big[d(ASx_{2n}, Tx_{2n-1}) \\
&\quad + d(Bx_{2n-1}, S^2x_{2n}) \big], d(S^2x_{2n}, Tx_{2n-1}) \Big\} \\
&\quad + q \max\big\{ d(ASx_{2n}, S^2x_{2n}), d(Bx_{2n-1}, Tx_{2n-1}) \big\} \\
&\quad + r \max\Big\{ d(ASx_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, S^2x_{2n}), \tfrac{1}{2}\big[d(ASx_{2n}, Bx_{2n-1}) \\
&\quad + d(Tx_{2n-1}, S^2x_{2n}) \big], d(S^2x_{2n}, Tx_{2n-1}) \Big\}
\end{aligned}
$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Sz,z) &\le p\max\left\{ d(Sz,Sz), d(z,z), \tfrac{1}{2}[d(Sz,z)+d(z,Sz)], d(Sz,z) \right\} + q\max(0) \\ &\quad + r\max\left\{ d(Sz,z), d(z,z), \tfrac{1}{2}[d(Sz,z)+d(z,Sz)], d(Sz,z) \right\} \\ &\le p\max\left\{ 0, 0, \tfrac{1}{2}[d(Sz,z)+d(z,Sz)], d(Sz,z) \right\} \\ &\quad + r\max\left\{ d(Sz,z), \tfrac{1}{2}[d(Sz,z)+d(z,Sz)], d(Sz,z) \right\} \end{aligned}$$

so that $z = Sz$. By (3.2), we also obtain

$$\begin{aligned} d(Az,Bx_{2n-1}) &\le p\max\left\{ d(Az,Sz), d(Bx_{2n-1},Tx_{2n-1}), \tfrac{1}{2}[d(Az,Tx_{2n-1}) + d(Bx_{2n-1},Sz)], d(Sz,Tx_{2n-1}) \right\} \\ &\quad + q\max\left\{ d(Az,Sz), d(Bx_{2n-1},Tx_{2n-1}) \right\} \\ &\quad + r\max\left\{ d(Az,Tx_{2n-1}), d(Bx_{2n-1},Sz), \tfrac{1}{2}[d(Az,Bx_{2n-1}) + d(Tx_{2n-1},Sz)], d(Sz,Tx_{2n-1}) \right\} \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Az,z) &\le p\max\left\{ d(Az,Sz), 0, \tfrac{1}{2}[d(Az,z)+d(z,Sz)], d(Sz,z) \right\} \\ &\quad + q\,d(Az,Sz) + r\max\left\{ d(Az,z), d(z,Sz), \tfrac{1}{2}[d(Az,z) + d(z,Sz)], d(Sz,z) \right\} \end{aligned}$$

so that $z = Az$. Since $A(X) \subset T(X)$, we have $z \in T(X)$ and hence there exists a point $u \in X$ such that $z = Az = Tu$.

$$\begin{aligned} d(z,Bu) &= d(Az,Bu) \\ &\le p\max\left\{ d(Az,Sz), d(Bu,Tu), \tfrac{1}{2}[d(Az,Tu)+d(Bu,Sz)], d(Sz,Tu) \right\} \\ &\quad + q\max\left\{ d(Az,Sz), d(Bu,Tu) \right\} + r\max\left\{ d(Az,Tu), d(Bu,z), \tfrac{1}{2}[d(Az,Bu)+d(Tu,Sz)], d(Sz,Tu) \right\} \\ &\le p\max\left\{ d(Bu,Tu), \tfrac{1}{2}[d(Az,Tu)+d(Bu,z)], d(Sz,Tu) \right\} \\ &\quad + q\,d(Bu,Tu) + r\max\left\{ d(Az,Tu), d(Bu,z), \tfrac{1}{2}[d(Az,Bu)+d(Tu,Sz)], d(Sz,Tu) \right\} \end{aligned}$$

which implies that $z = Bu$. Since B and T are compatible on X and $Tu = Bu = z$, we have $d(TBu, BTu) = 0$ by Lemma 2.6 and hence $Tz = TBu = BTu = Bz$. Moreover, by (3.2), we obtain

$$\begin{aligned} d(z,Tz) &= d(Az,Bz) \\ &\le p\max\left\{ 0, d(Bz,Tz), \tfrac{1}{2}[d(z,Tz)+d(Bz,z)], d(z,Tz) \right\} \\ &\quad + q\,d(Bz,Tz) + r\max\left\{ d(z,Tz), d(Bz,z), \tfrac{1}{2}[d(z,Bz)+d(Tz,z)], d(z,Tz) \right\} \end{aligned}$$

so that $z = Tz$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when T is continuous.

Next, suppose that A is continuous. Since A and S are compatible on X, it follows from Lemma 2.7 that

$$A^2x_{2n} \to Az,\ SAx_{2n} \to Az \text{ as } n \to \infty$$

By (3.2), we have

$$\begin{aligned} d(A^2x_{2n}, Bx_{2n-1}) \le\ & p\max\Big\{ d(A^2x_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), \tfrac{1}{2}\big[d(A^2x_{2n}, Tx_{2n-1}) \\ & + d(Bx_{2n-1}, SAx_{2n})\big], d(SAx_{2n}, Tx_{2n-1})\Big\} \\ & + q\max\big\{ d(A^2x_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1})\big\} \\ & + r\max\Big\{ d(A^2x_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, SAx_{2n}), \tfrac{1}{2}\big[d(A^2x_{2n}, Bx_{2n-1}) \\ & + d(Tx_{2n-1}, SAx_{2n})\big], d(SAx_{2n}, Tx_{2n-1})\Big\} \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Az, z) \le\ & p\max\Big\{0, 0, \tfrac{1}{2}\big[d(Az, z) + d(z, Az)\big], d(Az, z)\Big\} + r\max\Big\{d(Az, z), d(z, Az), \\ & \tfrac{1}{2}\big[d(Az, z), d(z, Az)\big], d(Az, z)\Big\} \end{aligned}$$

so that $z = Az$. Hence there exists a point $v \in X$ such that $z = Az = Tv$. By (3.2), we also obtain

$$\begin{aligned} d(A^2x_{2n}, Bv) \le\ & p\max\Big\{ d(A^2x_{2n}, SAx_{2n}), d(Bv, Tv), \tfrac{1}{2}\big[d(A^2x_{2n}, Tv) \\ & + d(Bv, SAx_{2n})\big], d(SAx_{2n}, Tv)\Big\} \\ & + q\max\big\{ d(A^2x_{2n}, SAx_{2n}), d(Bv, Tv)\big\} \\ & + r\max\Big\{ d(A^2x_{2n}, Tv), d(Bv, SAx_{2n}), \tfrac{1}{2}\big[d(A^2x_{2n}, Bv) \\ & + d(Tv, SAx_{2n})\big], d(SAx_{2n}, Tv)\Big\} \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(z, Bv) \le\ & p\max\Big\{0, d(Bv, Tv), \tfrac{1}{2}\big[d(Az, Tv) + d(Bv, z)\big], d(z, Tv)\Big\} + q\,d(Bv, Tv) \\ & + r\max\Big\{d(Az, Tv), d(Bv, z), \tfrac{1}{2}\big[d(Az, Bv) + d(Tv, z)\big], d(z, Tv)\Big\} \end{aligned}$$

which implies that $z = Bv$. Since B and T are compatible on X and $Tv = Bv = z$, we have $d(TBv, BTv) = 0$ by Lemma 2.6 and hence $Tz = TBv = BTv = Bz$. Moreover, by (3.2),

we have

$$\begin{aligned} d(Ax_{2n}, Bz) \le\ & p\max\Big\{ d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), \tfrac{1}{2}\big[d(Ax_{2n}, Tz) \\ & + d(Bz, Sx_{2n})\big], d(Sx_{2n}, Tz)\Big\} \\ & + q\max\big\{ d(Ax_{2n}, Sx_{2n}), d(Bz, Tz)\big\} \\ & + r\max\Big\{ d(Ax_{2n}, Tz), d(Bz, Sx_{2n}), \tfrac{1}{2}\big[d(Ax_{2n}, Bz) \\ & + d(Tz, Sx_{2n})\big], d(Sx_{2n}, Tz)\Big\} \end{aligned}$$

Letting $n \to \infty$, we have

$$d(z,Bz) \le p\max\left\{0, d(Bz,Tz), \frac{1}{2}[d(z,Tz)+d(Bz,z)], d(z,Tz)\right\} + q\,d(Bz,Tz) + r\max\left\{d(z,Tz), d(Bz,z), \frac{1}{2}[d(z,Bz)+d(Tz,z)], d(z,Tz)\right\}$$

so that $z = Bz$. Hence there exists a point $B(X) \subset S(X)$, there exists a point $w \in X$ such that $z = Bz = Sw$ and so, by (3.2)

$$\begin{aligned} d(Aw,z) &= d(Aw,Bz) \\ &\le p\max\left\{d(Aw,Sw), 0, \frac{1}{2}[d(Aw,z)+d(z,Sw)], d(Sw,z)\right\} + q\,d(Aw,Sw) \\ &\quad + r\max\left\{d(Aw,z), d(z,Sw), \frac{1}{2}[d(Aw,z)+d(z,Sw)], d(Sw,z)\right\} \end{aligned}$$

so that $Aw = z$. Since A and S are compatible on X and $Aw = Sw = z$, we have $d(SAw, ASw) = 0$ and hence $Sz = SAw = ASw = Az$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when B is continuous. Finally, it follows easily from (3.2) that z is a unique common fixed points of A, B, S and T. This completes the proof.

Corollary 3.3. Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (3.1), (3.5) and (3.6). Suppose that

$$\begin{aligned} d(Ax,By) &\le p\max\left\{d(Ax,Sx), d(By,Ty), \frac{1}{2}d(Ax,Ty), \frac{1}{2}d(By,Sx), d(Sx,Ty)\right\} \\ &\quad + q\max\{d(Ax,Sx), d(By,Ty)\} \\ &\quad + r\max\left\{d(Ax,Ty), d(By,Sx), \frac{1}{2}d(Ax,By), \frac{1}{2}d(Ty,Sx), d(Sx,Ty)\right\} \end{aligned}$$

for all $x, y \in X$, where $0 < p + q + 2r < 1$. Then A, B, S and T have a unique common fixed point in X.

IV. Examples

In this section, we give a example to illustrate our main theorems. The following example was shown by some authors ([1], [3], [7], [9], [15]). Here, we need that the condition (3.2) satisfy in Theorem 3.2. In the following example, we show the existence of a common fixed point of mappings which are compatible, but they are not weakly commuting and commuting.

Example 4.1. Let $X = [1, \infty)$ with the Euclidean metric d. Define the mappings $A, B, S, T : X \to X$ by

$$Ax = x^3, \quad Bx = x^2, \quad Sx = 2x^6 - 1, \quad Tx = 2x^4 - 1$$

for all $x \in X$, respectively. Now, since

$$\begin{aligned} d(Sx,Ty) &= 2|x^3 - y^2||x^3 + y^2| \\ &\ge 4d(Ax,By) \end{aligned}$$

for all $x, y \in X$, we obtain

$$\begin{aligned} d(Ax,By) &\le \frac{1}{4}d(Sx,Ty) \\ &\le \frac{1}{4}\max\left\{d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty)+d(By,Sx)], d(Sx,Ty)\right\} \\ &\quad + q\max\{d(Ax,Sx), d(By,Ty)\} \\ &\quad + r\max\left\{d(Ax,Ty), d(By,Sx), \frac{1}{2}[d(Ax,By)+d(Ty,Sx)], d(Sx,Ty)\right\} \end{aligned}$$

for all $x, y \in X$, where $0 < q + 2r < \frac{3}{4}$. Therefore, we see that the hypotheses of Theorem 3.2 except the (weak) commutativity of A and S are satisfied, but A, B, S and T have a unique common fixed point in X.

Now, we show that the condition (3.1) is necessary in Theorem 3.2.

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