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Duality for Semi- Infinite Multiobjective Fractional Programming Problems Involving Generalized (H_p, R)-Invexity

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ABSTRCT:- A semi-infinite multiobjective fractional programming problem is considered. Wolfe and Mond-Weir-type dual programs under generalized (H_p, r)- invexity are considered. Weak, strong and strict converse duality relations are established under generalized (H_p, r)- invexity conditions.

Keywords:- Semi-infinite multiobjective fractional programming, Hp -invexity, Wolfe type duality, Mond-Weir type duality, Sufficiency.

I. INTRODUCTION

Convexity plays a vital role in many aspects of mathematical programming including optimality conditions and duality theory. To relax convexity assumptions imposed on the functions involved, various generalized notions have been proposed. Preda [1] introduced the concept of generalized convexity, an extension of F-convexity defined by Hanson and Mond [2] and generalized convexity defined by Vial [3]. Gulati and Islam [4] derived sufficiency and duality results for efficient and properly efficient solutions of a multiobjective nonlinear programming problem under the assumptions taken by Hanson and Mond [2]. Ahmad [5] obtained a number of sufficiency theorems for efficient and properly efficient solutions under various generalized convexity assumptions for multiobjective programming problems.

Semi infinite optimization problems were introduced by Hettich and Kortanek [6]. Generalised semi infinite optimization problems were studied by Lopez and Still [7] and Vazquez and Ruckmann [8]. Avriel [9] first introduced the definition of *r*-convex functions and established some characterizations and the relations between *r*-convexity and other generalization of convexity. Antczak [10] introduced the concept of a class of r-preinvex functions, which is a generalization of r-convex functions and preinvex functions, and obtained some optimality results under r-preinvexity assumptions for constrained optimization problems. Antczak [11] introduced p-invex sets and (p,r)-invex functions and derived sufficient optimality conditions for a nonlinear programming problem involving (p, r)-invex functions. (p, r)-invex functions were further generlized as (H_p, r)-invex functions by Yuan [12]. Liu [13] obtained sufficient optimality conditions for multiple objective programming problem and multiobjective fractional programming problem involving (H_p, r)-invex functions. Jayswal et al. [14] established Generalized (H_p, r)-invexity in multiobjective programming problems. Jayswal et al. [15] introduced duality results for semi-infinite programming problems involving (H_p, r)-invex functions.
In this paper we have considered semi-infinite multiobjective fractional programming problem. Wolfe type and Mond-Weir type duals are considered, Weak, strong and strictly converse duality theorems are established by considering (H_p, r)-invexity conditions.

II. NOTATIONS AND PRELIMINARIES

Let R^n be the n-dimensional Euclidean space, $R^n_+ = \{x \in R^n \mid x$ □ $0\}$ and, $R^n_+ = \{x \in R^n \mid x > 0\}$. If x, y $\in R^n$, then x □ y is used to denote the case x_i □ y_i, i = 1, 2, … , n and $x \neq y$.
Antczak, T [11] introduced (p, r)-invex sets and (p, r)-invex functions as follows:
Definition 2.1: Let a_1, $a_2 > 0$, $\lambda \in (0,1)$ and $r \in R$. Then the weighted r-mean of a_1 and a_2 is given by

$$M_r\left(a_1,a_2,\lambda\right) = \begin{cases} \left(\lambda a_1^r + (1-\lambda)a_2^r\right)^{\frac{1}{r}}, & \text{for } r \neq 0, \\ a_1^{\lambda} a_1^{(1-\lambda)}, & \text{for } r = 0, \end{cases}$$

where $\lambda \in (0,1)$ and $r \in R$.
Definition 2.2 [12] : A subset $X \subset R^n$ is said to be H_p -invex set, if for any x, u $\in$ X, there exists a vector functions. $H_p: X \times X \times [0,1] \rightarrow R^n$, such that
$H_p(x, u; 0) = e^u$, $\quad H_p(x, u; \lambda) \in R^n_+$

In (H_p (x, u; λ) ∈ X, ∀ λ ∈ [0,1], p ∈ R.

In the above definitions, the logarithm and the exponentials appearing in the expressions are understood to be taken componentwise.

Throughout the paper, we assume that X be a H_p -invex set, H_p is right differentiable at 0 with respect to the variable λ for each given pair x, u ∈ X, and f : X → R is differential on X. The symbol H'_{p_1} (x, u; 0+) ≡ (H'_{p_1} (x, u; 0+),…, H'_{p_1} (x, u; 0+))^T denotes the right derivative of H_p at 0 with respect to the variable λ for each given pair x, u ∈ X; $\nabla f(x) \equiv (\nabla_1 f(x),...,$ $\nabla_n f(x))^T$ denotes the differential of f at x, and so $\frac{\nabla f(u)}{e^u}$ denotes $\left(\frac{\nabla_1 f(u)}{e^{u_1}},...,\frac{\nabla_n f(u)}{e^{u_n}}\right)^T$.

Liu, X et.al [13] introduced multiple objective programming involving differentiable (H_p, r)-invex functions as follows:

Definition 2.3: A differentiable function f : X → R is said to be (strictly) (H_p, r) – invex at u ∈ X, if for all x ∈ X, one of H_p' the relations

$$\frac{1}{r}\left[e^{r(f(x)-f(u))}-1\right] \geq \frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } (>) \text{ for } r \neq 0,$$

$$f(x)-f(u) \geq \frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } (>) \text{ for } r = 0,$$

hold.

If the above inequalities are satisfied at any point u ∈ X, then f is said to be (H_p, r)-invex (strictly (H_p, r)-invex) on X.

Jayswal et. al [14] introduced the generalized (H_p, r)-invex function as follows :

Definition 2.4: A differentiable function f : X → R is said to be (strictly) (H_p, r) -pseudo invex at u ∈ X, if for all x ∈ X, the relations

$$\frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } \geq 0 \Rightarrow \frac{1}{r}\left[e^{r(f(x)-f(u))}-1\right] \geq 0, \text{ for } r \neq 0,$$

$$\frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } \geq 0 \Rightarrow f(x)-f(u) \geq 0, \text{ for } r = 0,$$

hold.

If the above inequalities are satisfied at any point u ∈ X, then f is said to be (H_p, r)- pseudoinvex on x.

Definition 2.5 : A differentiable function f : X → R is said to be strict (H_p, r) -pseudoinvex at u ∈ X, if for all x ∈ X, the relations

$$\frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } \geq 0 \Rightarrow \frac{1}{r}\left[e^{r(f(x)-f(u))}-1\right] \geq 0, \text{ for } r \neq 0,$$

$$\frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } \geq 0 \Rightarrow f(x)-f(u) \geq 0, \text{ for } r = 0,$$

hold.

If the above inequalities are satisfied at any point u ∈ X, then f is said to be strict (H_p, r) -pseudoinvex on X.

Definition 2.6 : A differentiable function f : X → R is said to be (H_p, r)-quasiinvex at u ∈ X, if for all x ∈ X, the relations

$$\frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } \geq 0 \Rightarrow \frac{1}{r}\left[e^{r(f(x)-f(u))}-1\right] \geq 0, \text{ for } r \neq 0,$$

$$\frac{\nabla f(u)^T}{e^u} H_p' \text{ (x, u; 0+) } \geq 0 \Rightarrow f(x)-f(u) \geq 0, \text{ for } r = 0,$$

hold.

If the above inequalities are satisfied at any point u ∈ X, then f is said to be (H_p, r)- quasiinvex on X.

Remark 2.1 All the theorems in the subsequent parts of this paper will be proved only in the the case when r ≠ 0. The proofs in other case are easier than in this one. Also we assume that r > 0 (in the case when r < 0, the direction some of the inequalities in the proof of the theorems should be changed to the opposite one).
We consider the following semi-infinite programming (SIP) problem:

(SIP) $\underset{x\in R^n}{\text{Minimize}}\ f(x),$

subject to $h_j(x) \leq 0;\ \ j \in J$ (2.1)

where J is an index set which is possibly infinite, f and h_j, j ∈ J are differentiable functions from R^n to $R \cup \{+\infty\}$.
We consider the following semi-infinite multiobjective fractional programming (SIFP) problem:

(SIFP) Minimize $\left(\dfrac{f_1(x)}{g_1(x)}, \dfrac{f_2(x)}{g_2(x)}, \ldots, \dfrac{f_m(x)}{g_m(x)}\right)$

subject to $h_j(x) \leq 0,\ j \in J$ (2.2)

where, $f_i : X_0 \to R$, $g_i : X_0 \to R$, i = 1,2….m and $h_j : X_0 \to R$, j = 1,2…p, X_0 is an open subset of R^n. Also, $f_i(x) \geq 0$, $g_i(x) > 0$; i = 1,2….m.

3. First duality model

We consider the following Wolfe-type dual to (SIFP):

(WSIFD) Maximize $\sum_{i=1}^{m} \mu_i \left[f_i(u) - v_i g_i(u)\right] + \sum_{j=1}^{p} \lambda_j h_j(u)$

subject to

$$\sum_{i=1}^{m} \mu_i \left[\nabla f_i(u) - v_i \nabla g_i(u)\right] + \sum_{j=1}^{p} \lambda_j \nabla h_j(u) = 0 \quad (3.1)$$

where $\mu_i \geq 0, v_i \geq 0$ and $\lambda_j \geq 0$ and $\mu_i \neq 0$ and $v_i \neq 0$ for finitely many i∈ I, I is an index set which is possibly infinite, $\lambda_j \neq 0$ for finitely many j∈ J.

Theorem 3.1 (Weak duality):

Let x and (u, μ, v, λ), $\mu = (\mu_0)$, $\lambda = (\lambda_0)$, $v = (v_0)$, i ∈ I and j∈ J be feasible solution to (SIFP) and (WSIFD) respectively. Assume that $\sum_{i=1}^{m} \mu_i \left[f_i(.) - v_i g_i(.)\right] + \sum_{j=1}^{p} \lambda_j h_j(.)$ be (H_p, r)-invex at u. Then the following cannot hold:

$$\sum_{i=1}^{m} \mu_i \left[f_i(x) - v_i g_i(x)\right] < \sum_{i=1}^{m} \mu_i \left[f_i(u) - v_i g_i(u)\right] + \sum_{j=1}^{p} \lambda_j h_j(u)$$

Proof : On the contrary we assume that,

$$\sum_{i=1}^{m} \mu_i \left[f_i(x) - v_i g_i(x)\right] < \sum_{i=1}^{m} \mu_i \left[f_i(u) - v_i g_i(u)\right] + \sum_{j=1}^{p} \lambda_j h_j(u)$$

which together with the feasibility of x to (SIFP) gives

$$\sum_{i=1}^{m} \mu_i \left[f_i(x) - v_i g_i(x)\right] - \sum_{i=1}^{m} \mu_i \left[f_i(u) - v_i g_i(u)\right] + \sum_{j=1}^{p} \lambda_j h_j(x) - \sum_{j=1}^{p} \lambda_j h_j(u) < 0$$

Since r > 0, using the fundamental properties of exponential function, the above inequality yields

$$\frac{1}{r}\left[e^{r\left[\sum_{i=1}^{m} \mu_i [f_i(x) - v_i g_i(x)] - \sum_{i=1}^{m} \mu_i [f_i(u) - v_i g_i(u)] + \sum_{j=1}^{p} \lambda_j h_j(x) - \sum_{j=1}^{p} \lambda_j h_j(u)\right]} - 1\right] < 0.$$

The above inequality together with the assumption that $\sum_{i=1}^{m}\mu_i\left[f_i(.)\text{-}v_i g_i(.)\right]+\sum_{j=1}^{p}\lambda_j h_j(.)$ is (H_p, r)-invex at u, we obtain

$$\frac{\left[\sum_{i=1}^{m}\mu_i\left[\nabla f_i(u)\text{ - }v_i\nabla g_i(u)\right]+\sum_{j=1}^{p}\lambda_j\nabla h_j(u)\right]^T}{e^u}\,H_p'\ (x, u; 0+) < 0,$$

which contradicts (3.1). This completes the proof.

The proof of the following theorem is similar to Theorem 3.1 and hence being omitted.

Theorem 3.2 (Weak duality) : Let x and (u, μ, v, λ), $\mu = (\mu_i)$, $\lambda = (\lambda_j)$, $V = (V_i)$, $i \in I$ and $j \in J$ be feasible solution to (SIFP) and (WSIFD) respectively. Assume that $\sum_{i=1}^{m}\mu_i\left[f_i(.)\text{-}v_i g_i(.)\right]+\sum_{j=1}^{p}\lambda_j h_j(.)$ be (H_p, r)-pseudoinvex at u. Then the following cannot hold:

$$\sum_{i=1}^{m}\mu_i\left[f_i(x)\text{ - }v_i g_i(x)\right] < \sum_{i=1}^{m}\mu_i\left[f_i(u)\text{ - }v_i g_i(u)\right]+\sum_{j=1}^{p}\lambda_j h_j(u)$$

Theorem 3.3 (Strong duality): Let $\bar{x}$ be an optimal solution for (SIFP) and $\bar{x}$ satisfies a suitable constraints qualification for (SIFP). Then there exists $\bar{\mu}=\left(\bar{\mu_i}\right), \bar{\lambda}=\left(\bar{\lambda_j}\right), \bar{v}=\left(\bar{v_i}\right)$, $i \in I$ and $j\in J$ such that $\left(\bar{x},\bar{\mu},\bar{v},\bar{\lambda}\right)$ is feasible for (WSIFD). If any of the weak duality in Theorems 3.1 or 3.2 also holds, then $\left(\bar{x},\bar{\mu},\bar{v},\bar{\lambda}\right)$ is an optimal solution for (WSIFD).

Proof : Since $\bar{x}$ is optimal solution for (SIFP) and satisfy the suitable constraint qualification for (SIFP), then from Kuhn-Tucker necessary optimality condition there exists $\bar{\mu}=\left(\bar{\mu_i}\right)$, $\bar{\lambda}=\left(\bar{\lambda_j}\right), \bar{v}=\left(\bar{v_i}\right)$, $i \in I$ and $j \in J$ such that

$$\sum_{i=1}^{m}\bar{\mu_i}\left[\nabla f_i(\bar{x})\text{ - }v_i\nabla g_i(\bar{x})\right] + \sum_{j=1}^{p}\bar{\lambda_j}\nabla h_j(\bar{x}) = 0, \qquad \sum_{j=1}^{p}\bar{\lambda_j}h_j(\bar{x}) = 0,$$

which gives that the $\left(\bar{x},\bar{\mu},\bar{v},\bar{\lambda}\right)$ is feasible for (WSIFD).The optimality of $\left(\bar{x},\bar{\mu},\bar{v},\bar{\lambda}\right)$ for (WSIFD) follows from weak duality theorems. This completes the proof.

Theorem 3.4 (Strict converse duality) : Let $\bar{x}$ and $\left(\bar{y},\bar{\mu},\bar{v},\bar{\lambda}\right)$ be feasible solutions to (SIFP) and (WSIFD), respectively. Assume that $\sum_{i=1}^{m}\bar{\mu_i}\left[f_i(.)\text{-}\bar{v_i}g_i(.)\right]+\sum_{j=1}^{p}\bar{\lambda_j}h_j(.)$ is strictly (H_p, r)-invex at $\bar{y}$. Further assume that

$$\sum_{i=1}^{m}\bar{\mu_i}\left[f_i(\bar{x})\text{ - }\bar{v_i}g_i(\bar{x})\right]\le\sum_{i=1}^{m}\bar{\mu_i}\left[f_i(\bar{y})\text{ - }\bar{v_i}g_i(\bar{y})\right]+\sum_{j=1}^{p}\bar{\lambda_j}h_j(\bar{y}).$$

then $\bar{x} = \bar{y}$.

Proof : Let $\bar{x}$ be feasible solution to (SIFP) and $\left(\bar{y},\bar{\mu},\bar{v},\bar{\lambda}\right)$ be feasible to (WSIFD). Then

$$\sum_{i=1}^{m}\bar{\mu_i}\left[\nabla f_i(\bar{y})\text{ - }\bar{v_i}\nabla g_i(\bar{y})\right]+\sum_{j=1}^{p}\bar{\lambda_j}\nabla h_j(\bar{y}) = 0. \qquad (3.2)$$

Now, we assume that $\bar{x}\neq\bar{y}$ and exhibit a contradiction.

From the assumption that $\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(.)-\bar{v}_i g_i(.)\right]+\sum_{j=1}^{p}\bar{\lambda}_j h_j(.)$ is strictly (H_p, r)-invex at $\bar{y}$, we have

$$\frac{1}{r}\left[e^{r\left[\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{x})-\bar{v}_i g_i(\bar{x})\right]+\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{x})-\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{y})-\bar{v}_i g_i(\bar{y})\right]-\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{y})\right]}-1\right] >$$

$$\frac{\left[\sum_{i=1}^{m}\bar{\mu}_i\left[\nabla f_i(\bar{y})-\bar{v}_i\nabla g_i(\bar{y})\right]+\sum_{j=1}^{p}\bar{\lambda}_j\nabla h_j(\bar{y})\right]^T}{e^{\bar{y}}}\,H_p'(x, u; 0+),$$

which by the virtue of (3.2) becomes

$$\frac{1}{r}\left[e^{r\left[\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{x})-\bar{v}_i g_i(\bar{x})\right]+\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{x})-\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{y})-\bar{v}_i g_i(\bar{y})\right]-\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{y})\right]}-1\right] > 0.$$

As r > 0 , using the fundamental properties of exponential functions, we get

$$\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{x})-\bar{v}_i g_i(\bar{x})\right]+\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{x})-\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{y})-\bar{v}_i g_i(\bar{y})\right]-\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{y}) > 0.$$

From the feasibility of x to (SIFP), the above inequality yields

$$\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{x})-\bar{v}_i g_i(\bar{x})\right]>\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{y})-\bar{v}_i g_i(\bar{y})\right]+\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{y}),$$

which contradicts the assumption that

$$\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{x})-\bar{v}_i g_i(\bar{x})\right]\le\sum_{i=1}^{m}\bar{\mu}_i\left[f_i(\bar{y})-\bar{v}_i g_i(\bar{y})\right]+\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{y}).$$

Hence $\bar{x} = \bar{y}$. This completes the proof.

Next we consider the following Mond-Weir-type dual problem for (SIFP):

IV. SECOND DUALITY MODEL

(MWSIFD) Maximize $\left(\frac{f_1(u)}{g_1(u)},\frac{f_2(u)}{g_2(u)},\ldots,\frac{f_m(u)}{g_m(u)}\right)$

subject to

$$\sum_{i=1}^{m}\mu_i\left[\nabla f_i(u)-v_i\nabla g_i(u)\right]+\sum_{j=1}^{p}\lambda_j\nabla h_j(u)=0, \quad (4.1)$$

$$\sum_{i=1}^{m}\mu_i\left[f_i(u)-v_i g_i(u)\right]\ge 0, \quad (4.2)$$

$$\sum_{j=1}^{p}\lambda_j\nabla h_j(u) \ge 0, \quad (4.3)$$

where $\mu_i \ge 0$ and $\mu_i \ne 0$, $v_i \ge 0$ and $v_i \ne 0$, $\lambda_j \ge 0$ and $\lambda_j \ne 0$ for finitely many $i \in I$ and $j \in J$.

We have proved the following duality theorems.

Theorem 4.1 (Weak duality) : Let x and (u, μ, v, λ), $\mu = (\mu_i)$, $v = (v_i)$, $\lambda = (\lambda_j)$, $i \in I$ and j ∈ J, be feasible solution to (SIFP) and (MWSIFD), respectively. Assume that $\sum_{i=1}^{m}\mu_i\left[f_i(.) - v_i g_i(.)\right]$ and $\sum_{j=1}^{p}\lambda_j h_j(.)$ be (H_p, r) -invex at u. Then the following cannot hold:

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}.$$

Proof : Suppose contrary to the result, i.e.

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}.$$

Since r > 0, after some algebraic transformations, the above inequality yields

$$\frac{1}{r}\left[e^{r\left[\sum_{i=1}^{m}\mu_i\left[f_i(x) - v_i g_i(x)\right] - \sum_{i=1}^{m}\mu_i\left[f_i(u) - v_i g_i(u)\right]\right]} - 1\right] < 0.$$

From the assumption that $\sum_{i=1}^{m}\mu_i\left[f_i(.) - v_i g_i(.)\right]$ is (H_p, r) -pseudo invex at u.

$$\frac{\left[\sum_{i=1}^{m}\mu_i\left[\nabla f_i(u)-v_i\nabla g_i(u)\right]\right]^T}{e^u} H_p'(x, u; 0+) < 0. \qquad (4.4)$$

Since $\mu_i \geq 0, v_i \geq 0$ and $\lambda_j \geq 0$ and $\mu_i \neq 0$ and $v_i \neq 0$ for finitely many i∈ I, $\lambda_j \neq 0$ for finitely many j∈ J. from the feasibility of x and (u, μ, v, λ) to (SIFP) and (MWSIFD), respectively, we obtain

$$\sum_{j=1}^{p}\lambda_j h_j(x) \leq 0 \leq \sum_{j=1}^{p}\lambda_j h_j(u).$$

As r > 0, using the fundamental properties of exponential functions, we get

$$\frac{1}{r}\left[e^{r\left[\sum_{j=1}^{p}\lambda_j h_j(x) - \sum_{j=1}^{p}\lambda_j h_j(u)\right]} - 1\right] \leq 0,$$

which by the virtue of (H_p, r) -invexity of $\sum_{j=1}^{p}\lambda_j h_j(.)$ at u, gives

$$\frac{\left[\sum_{j=1}^{p}\lambda_j\nabla h_j(u)\right]^T}{e^u} H_p'(x, u; 0+) \leq 0, \qquad (4.5)$$

On adding (4.4) and (4.5) gives

$$\frac{\left[\sum_{i=1}^{m}\mu_i\left[\nabla f_i(u)-v_i\nabla g_i(u)\right]+\sum_{j=1}^{p}\lambda_j\nabla h_j(u)\right]^T}{e^u} H_p'(x, u; 0+) < 0,$$

which contradicts the dual constraint (4.1). This completes the proof.

The proof of the following theorem along the similar lines of Theorem 4.1, and hence being omitted.

Theorem 4.2 (Weak duality): Let x and (u, μ, v, λ), $\mu = (\mu_i)$, $v = (v_i)$, $\lambda = (\lambda_j)$, i ∈ I and j ∈ J, be feasible solutions to (SIFP) and (MWSIFD), respectively. Assume that $\sum_{i=1}^{m}\mu_i\left[f_i(.)\text{-}v_i g_i(.)\right]$ is (H_p, r)-pseudoinvex and $\sum_{j=1}^{p}\lambda_j h_j(.)$ is (H_p, r)-quasiinvex. Then the following cannot hold.

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}.$$

Theorem 4.3 (Strong duality): Let $\bar{x}$ be an optimal solution for (SIFP) and $\bar{x}$ satisfies a suitable constraints qualification for (SIFP). Then there exists $\bar{\mu} = \left(\overline{\mu_i}\right), \bar{\lambda} = \left(\overline{\lambda_j}\right), \bar{v} = \left(\overline{v_i}\right)$,

i ∈ I and j∈ J, such that $\left(\bar{x}, \bar{\mu}, \bar{v}, \bar{\lambda}\right)$ is feasible for (MWSIFD). If any of the weak duality in Theorems 4.1 or 4.2 also holds, then $\left(\bar{x}, \bar{\mu}, \bar{v}, \bar{\lambda}\right)$ is an optimal solution for (MWSIFD).

Proof: Since $\bar{x}$ is optimal solution for (SIFP) and satisfy the suitable constraint qualification for (SIFP), then from Kuhn-Tucker necessary optimality condition there exists $\bar{\mu} = \left(\overline{\mu_i}\right)$ $\bar{\lambda} = \left(\overline{\lambda_j}\right), \bar{v} = \left(\overline{v_i}\right)$, i ∈ I and j∈ J, such that

$$\sum_{i=1}^{m}\overline{\mu_i}\left[\nabla f_i(\bar{x}) - v_i\nabla g_i(\bar{x})\right] + \sum_{j=1}^{p}\overline{\lambda_j}\nabla h_j(\bar{x}) = 0, \qquad \sum_{j=1}^{p}\overline{\lambda_j}h_j(\bar{x}) = 0,$$

which gives that the $\left(\bar{x}, \bar{\mu}, \bar{v}, \bar{\lambda}\right)$ is feasible for (MWSIFD). The optimality of $\left(\bar{x}, \bar{\mu}, \bar{v}, \bar{\lambda}\right)$ for (MWSIFD) follows from weak duality theorems. This completes the proof.

Theorem 4.4 (Strict converse duality): Let $\bar{x}$ and $\left(\bar{y}, \bar{\mu}, \bar{v}, \bar{\lambda}\right)$ be a feasible solution to (SIFP) and (MWSIFD), respectively. Assume that $\sum_{i=1}^{m}\overline{\mu_i}\left[f_i(.) - \bar{v}_i g_i(.)\right]$ be strictly (H_p, r)-pseudoinvex and $\sum_{j=1}^{p}\overline{\lambda_j}h_j(.)$ be (H_p, r)-quasiinvex at $\bar{y}$. Further assume that

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} < \frac{f_i(\bar{y})}{g_i(\bar{y})}.$$

Then $\bar{x} = \bar{y}$. i.e. $\bar{y}$ is an efficient solution for (SIFP).

Proof : Let $\bar{x}$ be feasible solution to (SIFP) and $\left(\bar{y}, \bar{\mu}, \bar{v}, \bar{\lambda}\right)$ be feasible to (MWSIFD). Then

$$\sum_{i=1}^{m}\overline{\mu_i}\left[\nabla f_i(\bar{y}) - \overline{v_i}\nabla g_i(\bar{y})\right] + \sum_{j=1}^{p}\overline{\lambda_j}\nabla h_j(\bar{y}) = 0. \tag{4.6}$$

Now, we assume that $\bar{x} \neq \bar{y}$ and exhibit a contradiction.

Since $\mu_i \geq 0, v_i \geq 0$ and $\lambda_j \geq 0$ and $\mu_i \neq 0$ and $v_i \neq 0$ for finitely many i ∈ I, $\lambda_j \neq 0$ for finitely many j ∈ J. from the feasibility of $\bar{x}$ and $\left(\bar{y}, \bar{\mu}, \bar{v}, \bar{\lambda}\right)$ to (SIFP) and (MWSIFD), respectively, we obtain

$$\sum_{j=1}^{p}\overline{\lambda_j}h_j(\bar{x}) \leq 0 \leq \sum_{j=1}^{p}\overline{\lambda_j}h_j(\bar{y}),$$

As $r > 0$, using the fundamental properties of exponential functions, we get

$$\frac{1}{r}\left[e^{r\left(\sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{x}) - \sum_{j=1}^{p}\bar{\lambda}_j h_j(\bar{y})\right)} - 1\right] \leq 0$$

which by the virtue of (H_p, r)-quasiinvexity of $\sum_{j=1}^{p}\overline{\lambda_j} h_j(.)$ at $\bar{y}$, gives

$$\frac{\left[\sum_{j=1}^{p}\overline{\lambda_j}\,\nabla h_j(\bar{y})\right]^T}{e^{\bar{y}}} H_p'(\bar{x}, \bar{y}; 0+) \leq 0,$$

which along with (4.6) gives

$$\frac{\sum_{i=1}^{m}\overline{\mu_i}\left[\nabla f_i(\bar{y}) - \overline{v_i}\nabla g_i(\bar{y})\right]}{e^{\bar{y}}} H_p'(\bar{x}, \bar{y}; 0+) \geq 0,$$

From the above inequality together with the assumption that $\sum_{i=1}^{m}\mu_i\left[f_i(.) - v_i g_i(.)\right]$ is strictly

(H_p, r)-pseudoinvex at $\bar{y}$, we obtain

$$\frac{1}{r}\left[e^{r\left[\sum_{i=1}^{m}\overline{\mu_i}\left[f_i(\bar{x}) - \overline{v_i} g_i(\bar{x})\right] - \sum_{i=1}^{m}\overline{\mu_i}\left[f_i(\bar{y}) - \overline{v_i} g_i(\bar{y})\right]\right]} - 1\right] > 0,$$

which by the fundamental properties of exponential functions, yields

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} > \frac{f_i(\bar{y})}{g_i(\bar{y})}$$

which contradicts the fact that $\frac{f_i(\bar{x})}{g_i(\bar{x})} \leq \frac{f_i(\bar{y})}{g_i(\bar{y})}$. Hence $\bar{x} = \bar{y}$. This completes the proof.

V. CONCLUSIONS

Here we have used generalized (H_p, r)-invex functions and considered Wolfe and Mond-Weir type of dual programs for a class of semi-infinite multiobjective fractional programming problem and established the weak, strong and strict converse duality theorems assuming the functions involved to be generalized (H_p, r)-invex functions.

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