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Flip bifurcation and chaos control in discrete-time Prey-predator model

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Abstract:- The dynamics of discrete-time prey-predator model are investigated. The result indicates that the model undergo a flip bifurcation which found by using center manifold theorem and bifurcation theory. Numerical simulation not only illustrate our results, but also exhibit the complex dynamic behavior, such as the periodic doubling in period-2, -4 -8, quasi- periodic orbits and chaotic set. Finally, the feedback control method is used to stabilize chaotic orbits at an unstable interior point.

Keywords:- Discrete model, bifurcation theory, manifold theorem, feedback control.

I. INTRODUCTION

The mathematical modeling of ecological problems has a long and successful history. It is well known that the mathematical model has system of differential \difference equations that describing the dynamics of the interacting of species [1-3]. One of the following pioneering examples in mathematical ecology is the following continuous-time population model with two equations:

$$\begin{cases} \dot{x} = x(a - by - mx) \\ \dot{y} = y(-c + dx - ny) \end{cases} \qquad (1.1)$$

Where $x(t)$ and $y(t)$ are denoted to the population of two species at time t. If the parameters a, b, c, d, m and n are all positive then these differential equations are used to describe two species with limit growth called competing species [4-6]. If $m = n = 0$ with the positive parameters a, b, c and d then these differential equations called the predator-prey model of Volterra and Lotka, (see [7-9] and their references).

Obviously, if $a = m$ then the first equation in the model (1.1) represents the reproduction rate per individual. A different equation holds for the second population, so we can see the system of ordinary differential equations as in:

$$\begin{cases} \dot{x} = ax(1 - x) - bxy \\ \dot{y} = cy(1 - y) + bxy \end{cases} \qquad (1.2)$$

Where a, b and c are positive parameters such that a and c are the intrinsic growth rate of prey and predator densities, respectively. If $b = 0$ then each population growth expontialy [6].

By applying the forward Euler's scheme to the model (1.2) we obtain the discrete-time prey-predator model as follows:

$$\begin{cases} X_{n+1} = X_n + h[aX_n(1 - X_n) - bX_nY_n] \\ Y_{n+1} = Y_n + h[cY_n(1 - Y_n) + bX_nY_n] \end{cases} \qquad (1.3)$$

Where h is the step size.

The following is the organization of this paper: In the second section, we discuss the existence and local stability of possible fixed points of model(1.3). In the third section, we show that the model (1.3) undergo flip bifurcation with choosing h as a bifurcation parameter. In the next section, we present the numerical simulation which not only illustrate our results with theoretical analysis but also exhibit the complex dynamic behavior in the model(1.3). In the fifth section, the feedback control method is used to control chaotic orbits at an unstable positive fixed point. The conclusion is given in the last section.

II. ANALYSIS OF FIXED POINTS

In this section, the possible fixed points are obtained. The local stability conditions are organized and proposed with some propositions as follow:

Let the model (1.3) equations equal to the vector $(X, Y)^T$ then with simple computation we get the following fixed points:

1) $(X, Y) = (0,0)$ is the origin which always exists.
2) $(X, Y) = (1,0)$ is the first axial fixed point which means the prey population exist with absence of predator one.

3) $(X, Y) = (0,1)$ is the second axial fixed point which means the predator population exist with absence of prey one.

4) $(X, Y) = (x^*, y^*)$ is the unique positive fixed point which exist if and only if $a > b$, where $x^* = \frac{ac-bc}{ac+b^2}$ and $y^* = \frac{ac+ab}{ac+b^2}$.

In the model (1.3), we have got two axial fixed points (1,0) and (0,1) becouse each prey and predator population has no overlap between successive generations. So, their population evolves in discrete-time steps and can be models by the logistic equation [9].

Now, to study the stability of each fixed point we shall obtain the variation matrix and its characteristic equation. In general with (X, Y) fixed point we get the following Jacobian matrix:

$$J((X,Y)) = \begin{pmatrix} 1+h[a(1-2X)-bY] & -hbX \\ hbY & 1+h[c(1-2Y)+bX] \end{pmatrix} \quad (2.1)$$

And characteristic equation of $J((X, Y))$ is:

$$F(\lambda) = \lambda^2 + Tr(J((X,Y)))\lambda + Det(J((X,Y))) \quad (2.2)$$

Where

$Tr(J((X,Y))) = 2 + h[a(1-2X) + bX - bY]$

and

$Det(J((X,Y))) = 1 + h[a(1-2X) + c(1-2Y) + bX - bY] + h^2[a(1-2X) - bY][c(1-2Y) + bX]$

Hence the model (1.3) is a dissipative dynamical system if $|1 + h[a(1-2X) + c(1-2Y) + bX - bY] +$ *h2[a1−2X−bY][c1−2Y+bX]<1* [10].

The next propositions provide the local stability conditions near each fixed point with respect to the Lemma in [9].

Proposition 2.1: The origin fixed point (0,0) is source.

Obviously, the roots of the origin's characteristic equation are $\lambda_1 = 1 + ha$ and $\lambda_2 = 1 + hc$ which are both greater than one, i.e. $|\lambda_i| > 1$ for all $i = 1,2$.

Proposition 2.2: The prey axial fixed point (1,0) is saddle if $0 < h < \frac{2}{a}$, source $h > \frac{2}{a}$ and non-hyperbolic if $h = \frac{2}{a}$.

We can see that if $h = \frac{2}{a}$, one of the eigenvalues of the fixed point (1,0) is -1 and other is not one with module. Thus, the flip bifurcation may occur when the parameters vary in the neighborhood of $h = \frac{2}{a}$.

Proposition 2.3: There exist at least four different topological types of the predator axial fixed point (0,1) for all values of parameters, which means (0,1) is:

a) Sink if $a < b$ and $0 < h < \min\{\frac{2}{c}, \frac{2}{b-a}\}$;

b) Source if $a < b$ and $h > \max\{\frac{2}{c}, \frac{2}{b-a}\}$ (or $a > b$ and $h > \frac{2}{c}$);

c) Non-hyperbolic if $a = b$, $h = \frac{2}{c}$ or $h = \frac{2}{b-a}$;

d) Saddle otherwise.

We can easily see that one of the eigenvalues of the predator axial fixed point $(0,1)$ is -1 and other is neither 1 nor -1 if the topological type (3) of proposition (2.3) holdes. Thus, the fixed point (0,1) can undergo flip bifurcation because the system (1.3) restricted to logistic equation [8].

Proposition 2.4: if $a > b$ then the unique positive fixed point (x^*, y^*) is:

1. Sink if one of the following conditions holds:
A. $\Delta \geq 0$ and $0 < h < h_*$;
B. $\Delta < 0$ and $0 < h < h_{***}$;
2. Source if one of the following conditions holds:
A. $\Delta \geq 0$ and $h > h_{**}$;
B. $\Delta < 0$ and $h > h_{***}$;
3. Non-hyperbolic if one of the following conditions holds:
A. $\Delta \geq 0$ and $h = h_*$ or h_{**};
B. $\Delta < 0$ and $h = h_{***}$;
4. Saddle if the following conditions holds:

$\Delta \geq 0$ and $h_* < h < h_{**}$.

Where

$$h_* = \frac{ac(a + ac + ab - b) + \sqrt{\Delta}}{ac + b^2}$$

$$h_{**} = \frac{ac(a + ac + ab - b) - \sqrt{\Delta}}{ac + b^2}$$

$$h_{***} = \frac{(ac+b^2)(a+ac+ab-b)}{(a-b)(b+c)(a-b^2)}$$

and

$\Delta = a^2c^2(a+ac+ab-b)^2 - 4(a-b)(ac+ab)(a^2c^2+b^2c)$

From proposition(2.4), if condition(A) in(3)holds then one of the eigenvalues of the Jacobian matrix$J(x^*, y^*)$ is -1 and the other is neither 1 nor-1. We can rewrite condition(A) as the form$(a,b,c,h) \in M_1 \cap M_2$. Where $M_1 = \{(a,b,c,h): h = h_*, a > b, \Delta \geq 0, a,b,c > 0\}$ and $M_2 = \{(a,b,c,h): h = h_{**}, a > b, \Delta \geq 0, a,b,c > 0\}$.

In the following section we will see that there may be a flip bifurcation around the fixed (x^*, y^*) if h varies in the small neighborhood of h_* or h_{**} and$(a,b,c,h) \in M_1$ or$(a,b,c,h) \in M_2$.

III. BIFURCATION ANALYSIS

Based on the analysis on the section2, in this section we choose the step size h as bifurcation parameter to study the flip bifurcation of (x^*, y^*) by using the center manifold theorem and bifurcation theory [11, 12].

To do this we make h varies in the small neighborhood of h_* and$(a,b,c,h_*) \in M_1$. We can give similar argument for the case in which h varies in small neighborhood of h_{**} and$(a,b,c,h_{**}) \in M_2$.

Taking the parameters $(a,b,c,h_*) \in M_1$ arbitrarily, given a perturbation h^* of parameter h, we consider model(1.3) with perturbation h^* as follows:

$$\begin{cases} X_{n+1} = X_n + (h+h^*)[aX_n(1-X_n) - bX_nY_n] \\ Y_{n+1} = Y_n + (h+h^*)[cY_n(1-Y_n) + bX_nY_n] \end{cases} \qquad (3.1)$$

Where$|h^*| \ll 1$.

Let $U_n = X_n - X^*$ and$V_n = Y_n - Y^*$, then we transform the positive fixed point (x^*, y^*) of (3.1) into the origin. By calculating we obtained:

$$\begin{cases} U_{n+1} = U_n + (h+h^*)[a(U_n+X^*) - a(U_n+X^*)^2 - b(U_n+X^*)(V_n+Y^*)] \\ V_{n+1} = V_n + +(h+h^*)[c(V_n+Y^*) - c(V_n+Y^*)^2 + b(U_n+x^*)(V_n+Y^*)] \end{cases} \qquad (3.2)$$

Expanding model (3.2) as a Taylor series at $(U_n, V_n) = (0,0)$ to the second order, it becomes the following model:

$$\begin{cases} U_{n+1} = a_{11}U_n + a_{12}V_n + a_{13}U_nV_n + a_{14}{U_n}^2 + b_{11}h^*U_n + b_{12}h^*V_n + b_{13}h^*U_nV_n + b_{14}h^*{U_n}^2 \\ V_{n+1} = a_{21}U_n + a_{22}V_n + a_{23}U_nV_n + a_{24}{V_n}^2 + b_{21}h^*U_n + b_{22}h^*V_n + b_{23}h^*U_nV_n + b_{24}h^*{V_n}^2 \end{cases} \qquad (3.3)$$

Where

$a_{11} = 1 + h(a - 2ax^* - by^*)$, $a_{12} = -hbx^*$, $a_{13} = -hb$, $a_{14} = -ha$;

$b_{11} = \frac{a_{11}-1}{h}$, $b_{12} = \frac{a_{12}}{h}$, $b_{13} = \frac{a_{13}}{h}$, $b_{14} = \frac{a_{14}}{h}$;

$a_{21} = hby^*$, $a_{22} = 1 + h(c - 2cy^* + bx^*)$, $a_{23} = -a_{13}$, $a_{24} = -hc$;

$b_{21} = \frac{a_{21}}{h}$, $b_{22} = \frac{a_{22}-1}{h}$, $b_{23} = \frac{a_{23}}{h}$, $b_{24} = \frac{a_{24}}{h}$.

Let a matrix T is defined as follow:

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1-a_{11} & \lambda_2 - a_{11} \end{pmatrix}$$

Then the matrix T is invertible. Using transformation

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = T \begin{pmatrix} P_n \\ Q_n \end{pmatrix}$$

Then model (3.3) becomes of the following form:

$$\begin{cases} P_{n+1} = -P_n + F(U_n, V_n, h^*) + O(({U_n}^2, {V_n}^2)) \\ Q_{n+1} = \lambda_2 Q_n + G(U_n, V_n, h^*) + O(({U_n}^2, {V_n}^2)) \end{cases} \qquad (3.4)$$

Where

$$\begin{aligned} F(U_n, V_n, h^*) = {} & \frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]}{a_{12}(\lambda_2+1)} U_nV_n + \frac{a_{14}(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}{U_n}^2 - \frac{a_{24}}{(\lambda_2+1)}{V_n}^2 \\ & - \frac{[b_{11}(\lambda_2 - a_{11}) - b_{21}a_{12}]}{a_{12}(\lambda_2+1)} U_nh^* + \frac{[b_{12}(\lambda_2 - a_{11}) - b_{22}a_{12}]}{a_{12}(\lambda_2+1)} V_nh^* \\ & + \frac{[b_{13}(\lambda_2 - a_{11}) - b_{23}a_{12}]}{a_{12}(\lambda_2+1)} U_nV_nh^* + \frac{b_{14}(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}{U_n}^2h^* - \frac{b_{24}}{(\lambda_2+1)}{V_n}^2h^* \end{aligned}$$

$$\begin{aligned} G(U_n, V_n, h^*) = {} & \frac{[a_{13}(1 + a_{11}) + a_{12}a_{23}]}{a_{12}(\lambda_2+1)} U_nV_n + \frac{a_{14}(1 + a_{11})}{a_{12}(\lambda_2+1)}{U_n}^2 + \frac{a_{24}}{(\lambda_2+1)}{V_n}^2 \\ & + \frac{[b_{11}(1 + a_{11}) + b_{21}a_{12}]}{a_{12}(\lambda_2+1)} U_nh^* + \frac{[b_{12}(1 + a_{11}) + b_{22}a_{12}]}{a_{12}(\lambda_2+1)} V_nh^* \\ & + \frac{[b_{13}(1 + a_{11}) + b_{23}a_{12}]}{a_{12}(\lambda_2+1)} U_nV_nh^* + \frac{b_{14}(1 + a_{11})}{a_{12}(\lambda_2+1)}{U_n}^2h^* + \frac{b_{24}}{(\lambda_2+1)}{V_n}^2h^* \end{aligned}$$

Now, we determine the center manifold $w^c(0,0)$ of model (3.4) at the fixed point (0,0) in small neighborhood of $h^* = 0$. By the center manifold theory we can obtain the approximate representation of the center manifold $w^c(0,0)$ as follow:

$w^c(0,0) = \{(P_n, Q_n): Q_n = a_1 h^* + a_2 h^{*2} + a_3 P_n h^* + a_4 {P_n}^2 + O((|P_n| + |h^*|^2))\}$

Where

$a_1 = a_2 = 0,$

$a_3 = \frac{(1+a_{11})[b_{12}(1+a_{11})+b_{22}a_{12}]}{a_{12}(\lambda_2+1)^2} - \frac{b_{11}(1+a_{11})+b_{21}a_{12}}{(\lambda_2+1)^2},$

$a_4 = \frac{(1+a_{11})[b_{14}a_{12}+b_{24}(1+a_{11})-b_{13}(1+a_{11})-b_{23}a_{12}]}{1-{\lambda_2}^2}.$

Furthermore, we have

$$P_{n+1} = -P_n + F(U_n, V_n, h^*)$$
$$= -P_n + c_1 {P_n}^2 + c_2 P_n h^* + c_3 {P_n}^2 h^* + c_4 P_n h^{*2} + c_5 {P_n}^3 + O((|P_n| + |h^*|^3))$$

Where

$$c_1 = \frac{1}{\lambda_2 + 1}\{a_{12}a_{14}(\lambda_2 - a_{11}) - (1 + a_{11})[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}] - b_{24}(\lambda_2 - a_{11})^2\}$$

$$c_2 = \frac{1}{a_{12}(\lambda_2 + 1)}\{a_{12}[b_{11}(\lambda_2 - a_{11}) - b_{21}a_{12}] - [b_{12}(\lambda_2 - a_{11}) - b_{22}a_{12}](1 + a_{11})\}$$

$$c_3 = \frac{a_2}{a_{12}(\lambda_2 + 1)}\{a_{12}[a_{13}(\lambda_2 - a_{11}) - a_{23}a_{12}][\lambda_2 - 2a_{11} - 1] + {a_{12}}^2 a_{14}(1 + a_{11}) + 2a_{12}a_{24}[(1 + a_{11})(\lambda_2 + 1)]\} + \frac{a_1}{a_{12}(\lambda_2 + 1)}\{a_{12}[b_{11}(\lambda_2 - a_{11}) - b_{21}a_{12}] + {a_{12}}^2 b_{14}(\lambda_2 + 1) - [b_{12}(\lambda_2 - a_{11}) - b_{22}a_{12}](1 + a_{11}) - a_{12}b_{24}(1 + a_{11})^2\}$$

$$c_4 = \frac{a_3}{a_{12}(\lambda_2 + 1)}\{a_{12}[a_{13}(\lambda_2 - a_{11}) - a_{23}a_{12}][\lambda_2 - 2a_{11} - 1] + {a_{12}}^2 a_{14}(1 + a_{11}) + 2a_{12}a_{24}[(1 + a_{11})(\lambda_2 + 1)]\} + \frac{a_2}{a_{12}(\lambda_2 + 1)}\{a_{12}[b_{11}(\lambda_2 - a_{11}) - b_{21}a_{12}] - [b_{12}(\lambda_2 - a_{11}) - b_{22}a_{12}]\} = 0$$

$$c_5 = \frac{a_4}{a_{12}(\lambda_2 + 1)}\{[a_{13}(\lambda_2 - a_{11}) - a_{23}a_{12}][\lambda_2 - 2a_{11} - 1] + a_{12}a_{14}(1 + a_{11}) + 2a_{24}[(1 + a_{11})(\lambda_2 + 1)]\}$$

There for, when model(3.4) is restricted to the center manifold $w^c(0,0)$ we obtain the map G^* as follows:

$$G^*(P_n) = -P_n + c_1 {P_n}^2 + c_2 P_n h^* + c_3 {P_n}^2 h^* + c_4 P_n h^{*2} + c_5 {P_n}^3 + O((|P_n| + |h^*|^3)) \qquad \dots(3.5)$$

In order to undergo a flip bifurcation for a map, we require that two discriminatory quantities $\propto_1$ and $\propto_2$ *are not zero.*

Where

$$\propto_1 = \left(G^*_{P_n h^*} + \frac{1}{2}G^*_{h^*}G^*_{P_n P_n}\right)\Big|_{(0,0)} = c_2$$

$$\propto_2 = \left(\frac{1}{6}G^*_{P_n P_n P_n} + \left(\frac{1}{2}G^*_{h^*}G^*_{P_n P_n}\right)^2\right)\Big|_{(0,0)} = {c_1}^2 + c_5$$

There for by the above analysis and the theorem in [11], we obtain the following result.

Theorem 3.1: if $\alpha_2 \neq 0$ then model (3.2) undergo a flip bifurcation at the fixed point (x^*, y^*) when the parameter h^* varies in small neighborhood of the origin. Moreover, if $\alpha_2 > 0$(resp. $\alpha_2 < 0$), then the periodic point which bifurcation from (x^*, y^*) are stable (resp. unstable).

IV. NUMERICAL SIMULATION

In this paper, we give the bifurcation diagram and phase portraits of model (1.3) to confirm the above theoretical analysis and show the new interesting complex dynamical behaviors by using numerical simulation. We will choose $a = 2.5, b = 0.3, c = 0.2, (x_0, y_0) = (0.85, 0.55)$ and $h \in [1, 1.5]$ in model (1.3) as an example.

Due to above parameter values and the control parameter range we have got that $(x^*, y^*) = (0.7457, 2.1186), h_* = 3.2054, \propto_1 = -1.8403$ and $\propto_2 = 2.7747$. Obviously, we have $(a, b, c, h_*, \Delta) \in M_1$. The figures (1) and(2) will show the correctness of theorem3.1.

From figure(1), we will see that the fixed point $(x^*, y^*) = (0.7457, 2.1186)$ is stable for $h < 1.3782$ and loses its stability with $h = 1.3782$; when $h > 1.3782$ the periodic doubling and chaos will appear with increasing of$h$.

Due to bifurcation diagram (A) in figure (1), we see that the model (1.3) changes from stable to periodic doubling, period-4, -8, quasi-periodic then chaotic when $h = 1.03, 1.22, 1.37, 1.4, 1.448$ and1.47, respectively.

The last but not at least, noted that if the control parameter $h = 1.22$ then $\propto_2 = 2.7747 > 0$ which means the periodic double is stable due to the theorem (3.1) and this is good numerical example to confirm theoretical analysis part(see figure(3)).

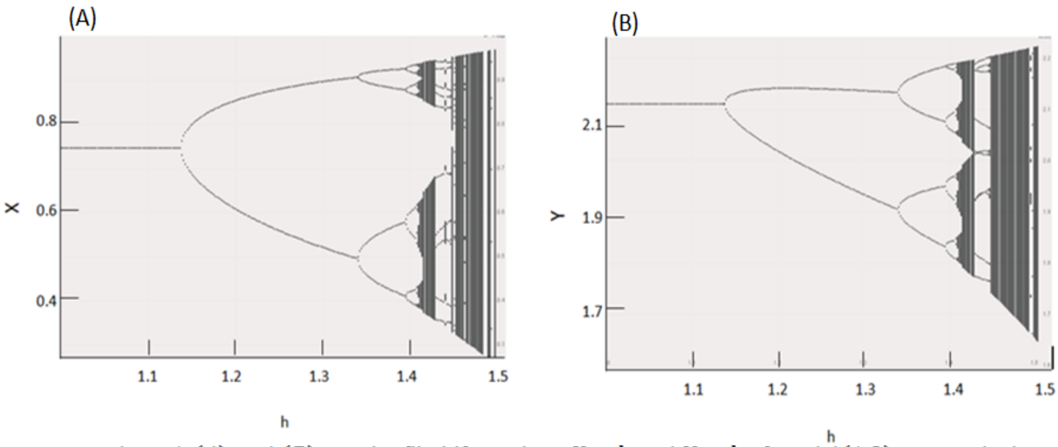

Figure1:(A) and (B) are the flip bifurcations $X - h$ and $Y - h$ of model(1.3), respectively.

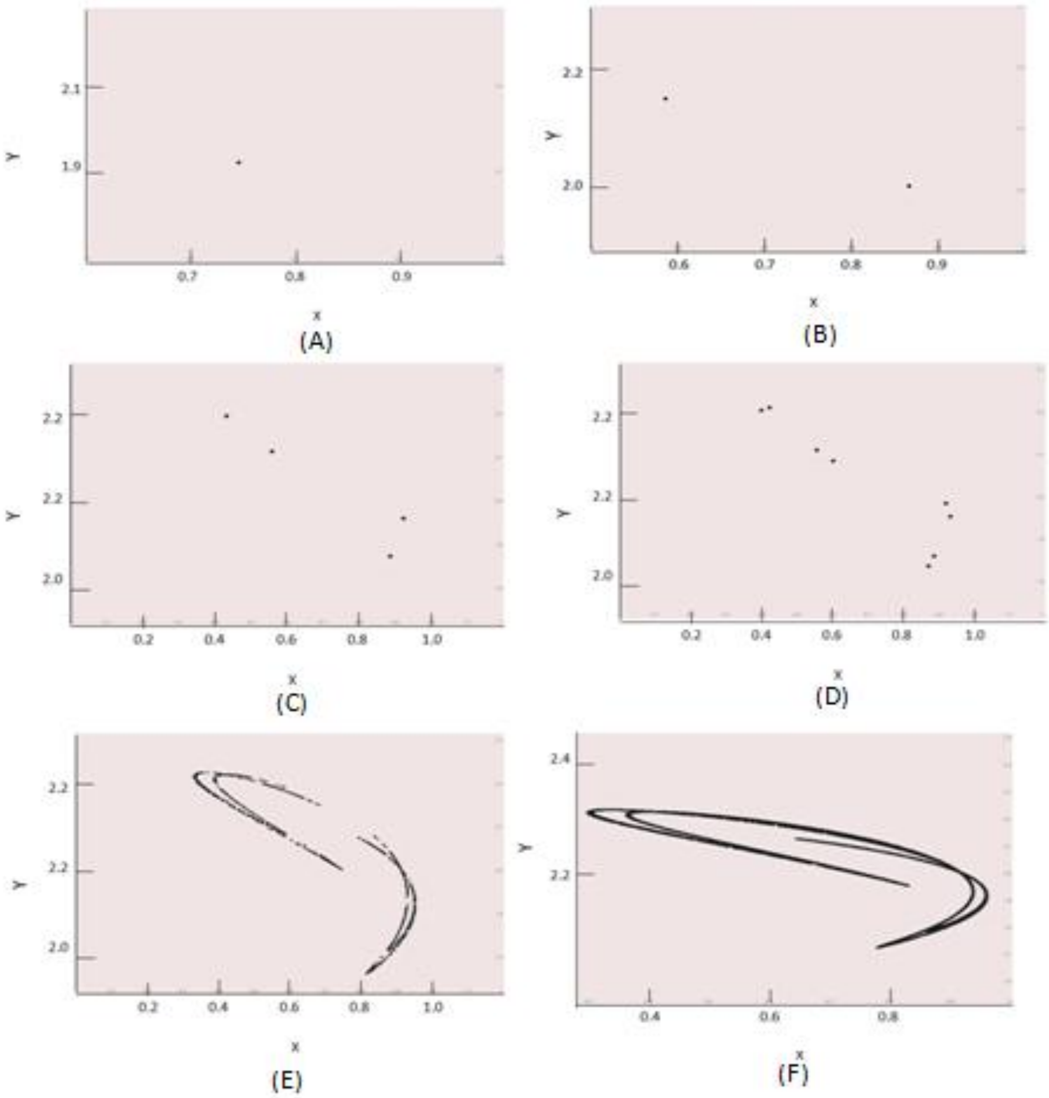

Figure2: Phase portrait corresponding to figure $2(A)$ here $h = 1.03, 1.22, 1.37, 1.4, 1.448$ and1.47, respectively.

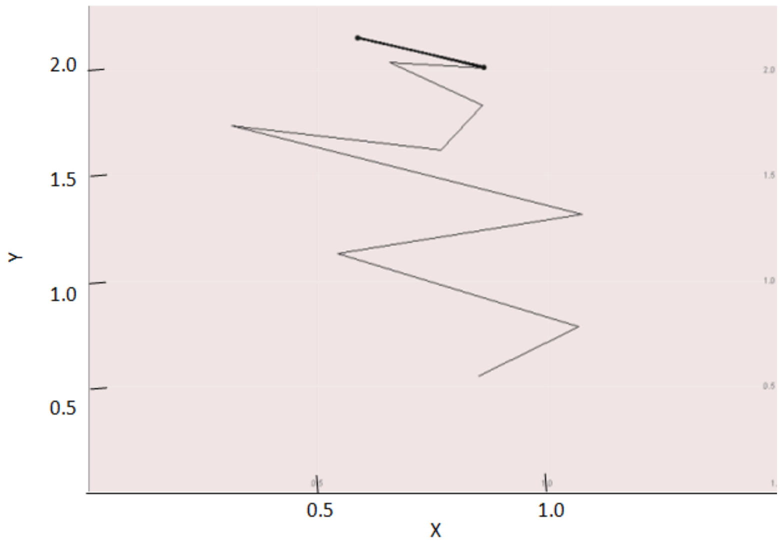

Figure3: The phase portrait of stable periodic double when $h = 1.22$

V. CONTROL CHAOS

In this section, the feedback control method [13-14] is used to stabilize chaotic orbits at an unstable positive fixed point of model(1.3).

Consider the following controlled form of model(1.3) :

$$\begin{cases} X_{n+1} = X_n + h[aX_n(1 - X_n) - bX_nY_n] + S_n \\ Y_{n+1} = Y_n + h[cY_n(1 - Y_n) + bX_nY_n] \end{cases} \quad (5.1)$$

with the following feedback control law as the control force:

$$S_n = -p_1(X_n - X^*) - p_2(Y_n - Y^*) \quad (5.2)$$

where $p_{1,2}$ is the feedback gain and (X^*, Y^*) is a positive fixed point of model(1.3).

The Jacobian matrix of model (5.1) at a fixed point (X^*, Y^*) is

$$J((X^*, Y^*)) = \begin{pmatrix} a_{11} - p_1 & a_{12} - p_2 \\ a_{21} & a_{22} \end{pmatrix}$$

where a_{11}, a_{12}, a_{21} and a_{22} are given in model (3.3).

The corresponding characteristic equation of matrix $J((X^*, Y^*))$ is:

$$\lambda^2 - (a_{11} + a_{22} - p_1)\lambda + a_{22}(a_{11} - p_1) - a_{21}(a_{12} - p_2) = 0 \quad (5.3)$$

Let $\lambda_{1,2}$ are the eigenvalues of (5.3) , then

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} - p_1 \quad (5.4)$$

and

$$\lambda_1\lambda_2 = a_{22}(a_{11} - p_1) - a_{21}(a_{12} - p_2) \quad (5.5)$$

The lines of marginal stability are determined by solving the equation$\lambda_1 = \pm 1$ and$\lambda_1\lambda_2 = 1$. These conditions guarantee that the eigenvalues λ_1 and λ_2 have modulus less than 1.

Suppose$\lambda_1\lambda_2 = 1$; from(5.5) we have line l_1 as follows:

$$a_{22}p_1 - a_{21}p_2 = a_{11}a_{22} - 1 \quad (5.6)$$

Suppose$\lambda_1 = 1, -1$; from(5.5) and (5.6) we have lines l_2 and $l_3$as follows:

$$(1 - a_{22})p_1 + a_{21}p_2 = a_{11} + a_{22} - 1 - a_{11}a_{22} + a_{12}a_{21} \quad (5.7)$$

and

$$(1 + a_{22})p_1 - a_{21}p_2 = a_{11} + a_{22} + 1 + a_{11}a_{22} - a_{12}a_{21} \quad (5.8)$$

The stable eigenvalues lie within a triangular region by line l_1 l_2 and l_3 .Therefore, some numerical simulations can be made to control the unstable fixed point (X^*, Y^*) by the state feedback method.

The parameters are selected as $a = 0.79, b = 0.3, c = 0.2, h = 4$, $(X, Y) = (0.85, 0.55)$ and the feedback gain $p_1 = -1.17502, p_2 = 0.18334$. A chaotic trajectory is stabilized at the fixed point $(0.395161, 1.592742)$ (see Figure (4)).

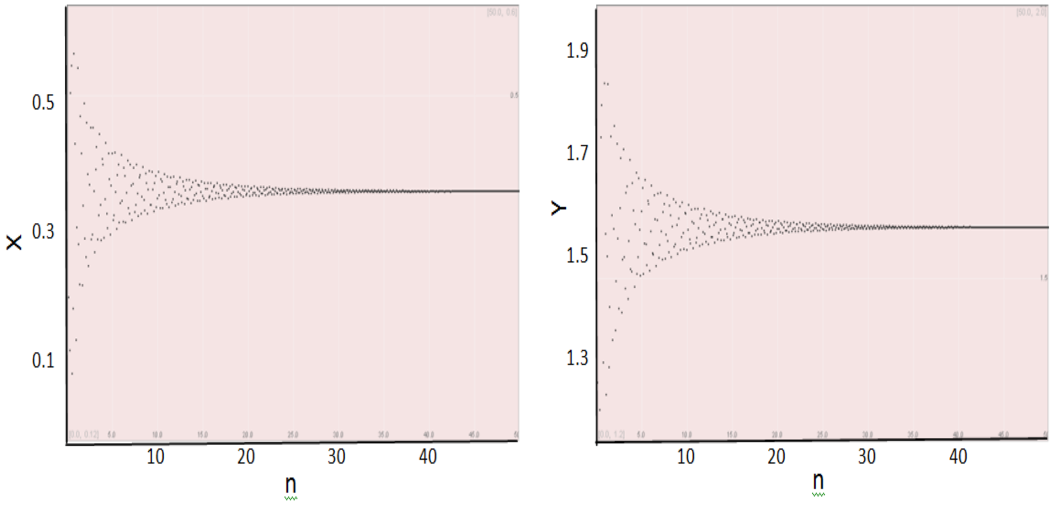

Figure 4: The time responses for the state X, Y of the controlled model (5.1) in the (n, X), (n, Y) plane.

VI. CONCLUSION

In this paper, the dynamical behavior of model (1.3) is discussed. If$a > b$, then the unique positive fixed point (x^*, y^*) arise beside the fixed points $(0,0), (1,0)$ and$(0,1)$. The model (1.3) exhibit complex and interesting dynamical behavior when the control parameter h is varying. That is if $(a, b, c, h) \in M_1$or M_2 and taking h as bifurcation parameter, then the flip bifurcation appears for the model(1.3). Moreover, the chaos control in model (1.3) is obtained.

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