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Maximal Ordered Leftideals in Ordered Ternarysemigroups.

Bindu. P[1], Dr.Sarala.Y[2], Dr. Madhusudhana Rao. D [3]
[1]*Research Scholar, Department of Mathematics, KL University, A.P, India.*
[2]*Faculty of Mathematics, Department of Mathematics, KL University, A.P, India.*
[3]*Faculty of Mathematics, Department of Mathematics, VSR & NVR College, Tenali, A.P*

Abstract:- Lehmer [4] gave the definition of a ternarysemigroup in 1932. We see that all semigroups can be converted in to a ternarysemigroup. In this paper, we give some axillary results which are also essential for our deliberations and characterize the relationship b/w the maximal ordered leftideals and the left simple and left (0)–simple ordered ternarysemigroups, Correspondent to the characterizations of maximal leftideals in ordered semigroups studied by Cao and Xu [2]

Keywords:- Maximal ordered leftideal and left (0 -) simple ordered ternarysemigroup.

I. INTRODUCTION AND PRELIMINARIES

Dixit and Dewan [3] initiate and considered the possessions of (Quasi , bi , left , right) leftideals in ternarysemigroups in 1995. Cao and Xu [2] characterized the maximal and minimal leftideals in ordered semigroups and gave some characterizations of maximal and minimal leftideals in ordered semigroupsin 2000. Arslanov and Kehayopulu [1] characterized the maximal and minimal ideals in ordered semigroups in 2002.

The idea of the maximal and minimal (left) ideals is the properly interested and important thing about ordered semigroups. Now we characterize the {0} – maximal ordered left ideals in ordered ternarysemigroups and give some characterizations of the (0-) maximal ordered left ideals in ordered ternarysemigroups Correspondent to the characterizations of the maximal and minimal left ideals in ordered semigroups considered by Cao and Xu.
In this paper our aim is four fold.

(1) To give the definition of an ordered ternarysemigroup.
(2) To initiate the idea of left simple & left (0) – simple ordered ternarysemigroups.
(3) To characterize the possessions of ordered left ideals in ordered ternarysemigroups.
(4) To characterize the relationship b/w the left simple & left (0)-simple ordered ternarysemigroups & maximal ordered leftideals.

It is important to know the definition of a ternarysemigroup to present the main theorems.

A non-empty set P is known as a ternarysemigroup [3] if ∃ a ternary operation P × P × P → P, written as $(x_1, x_2, x_3) \to [x_1 x_2 x_3]$, satisfying the adopting identity for any $x_1, x_2, x_3, x_4, x_5 \in$ P.
$[[x_1 x_2 x_3] x_4 x_5] = [x_1 [x_2 x_3 x_4] x_5] = [x_1 x_2 [x_3 x_4 x_5]]$.
Hence we see that all semigroups can be considered as a ternarysemigroups. For non-empty subsets A , B & C of a ternarysemigroup P, let [ABC] = $\{[abc] : a \in A, b \in B \text{ and } c \in C\}$
If A = $\{a\}$, then we write [$\{a\}$BC] as [aBC].
A ternarysemigroup P has a non-empty subset S is known as to be a ternary subsemigroup [3] of P if [SSS] $\subseteq$ S.
Example.1([3]): Let P = {-i , o , i}. Then P is a ternarysemigroup under the multiplication over Complex number while P is not a semigroup under complex number multiplication.
Example.2([3]): Let O = $\begin{pmatrix}0&0\\0&0\end{pmatrix}$, I = $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, A_1 = $\begin{pmatrix}1&0\\0&0\end{pmatrix}$, A_2 = $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, A_3 = $\begin{pmatrix}0&0\\1&0\end{pmatrix}$,
A_4 = $\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Then P={O, I, A_1, A_2, A_3, A_4} is a ternarysemigroup under matrix multiplication.

A partially ordered ternarysemigroup P is known as to be an ordered ternarysemigroup if for any $x_1, x_2, x_3, x_4 \in$ P, $x_1 \leq x_2 \Rightarrow [x_1 x_3 x_4] \leq [x_2 x_3 x_4]$ & $[x_4 x_3 x_1] \leq [x_4 x_3 x_2]$ if (P,.,≤) is an ordered ternarysemigroup and S is a ternary subsemigroup of P, then (S ,.,≤) is an ordered ternarysemigroup. For a subset H of an ordered ternarysemigroup P, we indicate
(H] = { $p \in$ P : $p \leq h$ for some $h \in$ H } & H ∪ a = H ∪ {a}, ∀ $a \in$ P. If H = {a}, we also write ({a}] as (a]. We see that H⊆ (H] , ((H]] = (H]. For subsets A and B of an ordered ternarysemigroups P, we have (A] ⊆ (B] if A ⊆ B and (A∪ B] = (A]∪ (B].

A non-empty subset L of an ordered ternanrysemigroup P is known as to be a leftideal of P if [PPL] $\subseteq$ L. A leftideal L of an ordered ternarysemigroup P is known as to be an ordered leftideal of P if for any $b \in$ P and $a \in$ L, $b \leq a \Rightarrow b \in$ L. The intersection of all ordered leftideals of a ternary subsemigroup S of an ordered ternarysemigroup P accommodating a non-empty subset A of S is the ordered lefideal of S generated by A. For A = {a}. Let $L_s(a)$ indicate the ordered leftideal of S generated by {a}. If S = P, Then we also write $L_P(a)$ as L(a). An element a of an ordered ternarysemigroup P with atleast two elements is known as to be a zero element of P if $[a\, p_1\, p_2] = [p_1\, a\, p_2] = [p_1 p_2\, a] = a$, $\forall\, p_1, p_2 \in$ P & $ab \leq p, \forall\, p \in$ P and we indicate it by 0. If P is an ordered ternarysemigroup with zero, then every ordered leftideal of P accommodates a zero element. An ordered ternarysemigroup P without zero is known as to be left simple if it has no proper ordered leftideals. An ordered ternarysemigroup P with zero is known as to be left 0 - simple if it has no nonzero proper ordered leftideals and [PPP] $\neq$ {0}. We shall give an example of an ordered ternarysemigroup without zero which there exists a ternary subsemigroup with zero.

***Example3*:**Let Z be the set of all integer. Define multiplication on Z by $[xyz] = \min\{x, y, z\}$, $\forall\, x, y, z \in$ Z. Then Z is an ordered ternarysemigrouop without zero under usual partial order. Let N be the set of all +ve integers. Then N is a ternary subsemigroup of Z with a zero element 1.

For any +ve integers m & n with m $\leq$ n and any elements $x_1, x_2, \ldots x_{2n}, x_{2n+1}$ of a ternarysemigroup P [5], we can write $[\, x_1 x_2 \ldots x_{2n} x_{2n+1} \,] = [\, x_1 \ldots x_m x_{m+1} \ldots x_{2n+1} \,]$
$= [\, x_1 \ldots [\, x_m x_{m+1} x_{m+2} \,]\, x_{m+3}\, x_{m+4} \,] \ldots x_{2n+1} \,]$. We shall assume that P stands for an ordered ternarysemigroup throughout this paper.

The adopting two lemmas are also essential for our deliberation and easy to verify.

***Lemma1.1*:**For any non-empty subset A of P , ([PPA]∪A] is the smallest ordered leftideal of P accommodating A. Furthermore, for any $a \in$ P, L (a) = ([PPa]∪ a].

***Lemma1.2*:**For any non-empty subset A of P, ([PPA]] is an ordered leftideal of P.

***Lemma1.3*:**If P has no zero element, then the adopting statements are equivalent.

(a) P is left simple. (b) ([PPa]] = P, $\forall\, a \in$ P. (c) L (a) = P, $\forall\, a \in$ P.

***Proof*:** By Lemma1.2 & P is left simple, we have ([PPa]] = P, $\forall\, a \in$ P.
Therefore (a) ⇒ (b).
By lemma1.1, we have L (a) = ([PPa] ∪ a] = ([PPa]] ∪ (a] = P∪(a] = P.
Thus (b) ⇒ (c).

Now let L be an ordered leftideal of P and let $a \in$ L. Then P = L (a) $\subseteq$ L $\subseteq$ P so L = P. Hence P is left simple, we have that (c) ⇒ (a). The proof is hence completed.

***Lemma1.4*:**If P has a zero element, then the adopting statements hold.

(a) If P is left 0 – simple, then L (a) = P, $\forall\, a \in$ P \ {a}.

(b) If L (a) = P, $\forall\, a \in$ P \ {0} then either [PPP] = {0} or P is left 0 – simple.

***Proof*:** Assume that P is left 0 – simple. Then L (a) is a non-zero ordered leftideal of P, $\forall\, a \in$ P\{0}. Hence L (a) = P, $\forall\, a \in$ P \{0}.

(b) Assume that L(a) = P, $\forall\, a \in$ P \{0} and let [PPP] $\neq$ {0}. Now let L be a non-zero ordered leftideal of P and put $a \in$ L\{0}. Then P = L (a) $\subseteq$ L $\subseteq$ P, so L = P.
Therefore P is left 0 – simple. Therefore the lemma proof is completed. The next lemma is easy to verify.

***Lemma1.5* :**Let {L_r : r $\in$ P} be a family of ordered leftideals of P. Then $\bigcup_{r \in P} L_r$ is an ordered leftideal of P and $\bigcap_{r \in P} L_r$ is also an ordered leftideal of P if $\bigcap_{r \in P} L_r \neq \phi$.

***Lemma1.6* :** If L is an ordered leftideal of P and S is a ternary subsemigroup of P, then the adopting statements hold.

(a) If S is left simple ∋ S $\cap$ L $\neq \phi$ then S $\subseteq$ L.

(b) If S is left 0 - simple ∋ S\{0} $\cap$ L $\neq \phi$ then S $\subseteq$ L.

***Proof* :**(a) Assume that S is left simple ∋ S $\cap$ L $\neq \phi$. Then let $a \in$ S $\cap$ L. Since L is an ordered leftideal of P, (L] $\subseteq$ L. By Lemma1.2, we have ([SSa]] $\cap$ S is an ordered leftideal of S. This implies that ([SSa]] $\cap$ S = S. Hence S $\subseteq$ ([SSa]) $\subseteq$ ([PPL]] $\subseteq$ (L] $\subseteq$ L, so S $\subseteq$ L.

(b) Assume that S is left 0 - simple such that S \ { 0 } $\cap$ L $\neq \phi$. Then let $a \in$ S \ { 0 } $\cap$ L.
By Lemma1.1 & 1.4 (a), we have S = $L_s(a)$ = ([SSa] $\cup$ a] $\cap$ S $\subseteq$ ([SSa] $\cup$ a] $\subseteq$ ([PPa] $\cup$ a]=L(a) $\subseteq$ L.Therefore S $\subseteq$ L. Hence the proof of the lemma is completed.

II. MAXIMAL ORDERED LEFTIDEALS

A proper ordered leftideal L of P is known as to be a maximal ordered leftideal of P if for any ordered leftideal A of P $\ni$ L $\subset$ A , we have A = P. Equivalently if for any proper ordered leftideal A of P $\ni$ L $\subseteq$ A, we have A = L.

In this part, we characterized the relationship b/w the maximality of ordered leftideals and the union $\mho$ of all (non- zero) proper ordered leftideals in ordered ternarysemigroups.

***Theorem2.1*:** If P has no zero element but it has proper ordered leftideals, then every proper ordered leftideals of P is maximal $\Leftrightarrow$ P consists of exactly one proper ordered leftideal or P consists of exactly two proper ordered leftideals L_1 & L_2, $L_1 \cup L_2$ = P & $L_1 \cap L_2 = \phi$.

***Proof*:** Consider that every proper ordered leftideal of P is maximal. Now assume L be a proper ordered leftideal of P. Then L is a maximal ordered leftideal of P. We study the adopting two cases.

Case *a*) P = L(x), $\forall\, x \in$ P \L.

If M is also a proper ordered leftideal of P and M $\neq$ L, then M is a maximal ordered leftideal of P. This implies that M \ L $\neq \phi$, so $\exists\, x \in$ M \L $\subseteq$ P \L. Thus P = L(x) $\subseteq$ M $\subseteq$ P. So M= P. It is not possible, so M =L. In this case, L is the unique proper ordered leftideal of P.

Case *b*) $\exists\, x \in$ P \L $\ni$ P $\neq$ L(x).

Then L(x) $\neq$ L & L(x) is a maximal ordered leftideal of P. By lemma1.5, L(x) $\cup$ L is an ordered leftideal of P. Since L $\subset$ L(x) $\cup$ L & L is a maximal ordered leftideal of P, L(x) $\cup$ L= P. By hypothesis and L(x) $\cap$ L $\subset$ L(x), we get L(x) $\cap$ L = ϕ. Now assume M be an arbitrary proper ordered leftideal of P. Then M is a maximal ordered leftideal of P. We note that M = M $\cap$ P = M $\cap$ (L(x) $\cup$ L) = (M $\cap$ L(x)) $\cup$ (M $\cap$ L). If (M $\cap$ L) $\neq \phi$, then M = L because M $\cap$ L & L are maximal ordered leftideals of P. If M $\cap$ L(x) $\neq \phi$, then M = L(x) because M $\cap$ L(x) & L(x) are maximal ordered leftideals of P . In this case, P consists of precisely two proper ordered leftideals L & L(x), L(x) $\cup$ L = P & L(x) $\cap$ L = ϕ. The reverse is apparent. By using the proof of Theorem2.1, we prove Theorem2.2.

***Theorem2.2*:** If P has a zero element and non-zero proper ordered leftideals, then every non- zero proper ordered leftideal of P is maximal $\Leftrightarrow$ P consists of precisely one non-zero proper ordered leftideal or P consists of precisely two non-zero proper ordered leftideals L_1& L_2, $L_1 \cup L_2$ = P & $L_1 \cap L_2$ = {0}.

***Theorem2.3*:** A proper ordered leftideal L of P is maximal $\Leftrightarrow$

1) P \L = {x} and ([PPx]] $\subseteq$ L forsome $x \in$ P (or)

2) P \L $\subseteq$ ([PPx]] , $\forall\, x \in$ P \L

***Proof*:** Consider that L is a maximal ordered leftideal of P. Then we study the adopting two cases.

Case *a*) $\exists\, x \in$ P \L $\ni$ ([PPx]] $\subseteq$ L. Then ([PPx]] $\subseteq$ L. By lemma1.1 we have L $\cup$ (x] = (L $\cup$ [PPx]] $\cup$ (x]= L $\cup$ (([PPx]] $\cup$ (x]) = L $\cup$ ([PPx] $\cup$ x] = L $\cup$ L(x). Thus L $\cup$ (x] is an ordered leftideal of P because L $\cup$ L(x) is an ordered leftideal of P. Since L is a maximal ordered leftideal of P and L $\subset$ L $\cup$ (x], we have L $\cup$ (x] = P. Hence P\L $\subseteq$ (x]. To show that P\L = {x}, assume $y \in$ P\L. Then $y \leq x$, so ([PPy]] $\subseteq$ ([PPx]] $\subseteq$ L. From ([PPy]] $\subseteq$ L and $y \in$ P\L, a corresponding argument presents that P\L $\subseteq$ (y]. Accordingly $x \leq y$, so y x. Hence P\L = {x}. In this case, the first condition is fulfilled.

Case *b*) ([PPx]] $\not\subseteq$ L , $\forall\, x \in$ P \L. If $x \in$ P \L, then ([PPx]] $\not\subseteq$ L and ([PPx]] is an ordered leftideal of P by Lemma1.2. By Lemma1.5, we have L $\cup$ ([PPx]] is an ordered leftideal of P and L $\subset$ L $\cup$ ([PPx]]. Since L is a maximal ordered leftideal of P, L $\cup$ ([PPx]] = P. Hence we conclude that P\L $\subseteq$ ([PPx]] , $\forall\, x \in$ P \ L. In this case the second condition is fulfilled.

Reversely, assume that K is an ordered leftideal of P such that L $\subset$ K. Then K\L $\neq \phi$. If P\L = {x} and ([PPx]] $\subseteq$ L for some $x \in$ P, then K \L $\subseteq$ P \L = {x}. Thus K\L = {x}, so K = L $\cup$ {x} = P. Hence L is a maximal ordered leftideal of P. If P\L $\subseteq$ ([PPx]], $\forall\, x \in$ P \L, then P \L $\subseteq$ ([PPy]] $\subseteq$ ([PPK]] $\subseteq$ K, $\forall$ $y \in$ K\L. Hence P = P \L $\cup$ L $\subseteq$ K $\subseteq$ P, so K = P. Therefore L is a maximal ordered leftideal of P. Now the theorem completed.

For an ordered ternarysemigroup P, assume $\mho$ indicate the union of all non-zero proper ordered leftideals of P if P has a zero element and assume $\mho$ indicate the union of all proper ordered leftideals of P if P has no zero element. Then it is easy to verify lemma2.4.

***Lemma2.4*:** $\mho$ = P $\Leftrightarrow$ L (x) $\neq$ P, $\forall\, x \in$ P.

According to Theorem2.3 & lemma2.4, we acquire the next two theorems.

***Theorem2.5*:** If P has no zero element, then one and only one of the adopting four conditions is fulfilled.

(1) P is left simple.

(2) L (x) $\neq$P, $\forall\, x \in$ P .

(3) $\exists\, x \in P \ni L(x) = P$, $x \notin ([PPx]]$ & $([PPx]] \subseteq \mho = P \setminus \{x\}$ and $\mho$ is the unique maximal ordered leftideal of P.
(4) $P \setminus \mho = \{y \in P : ([PPy]] = P\}$ and $\mho$ is the unique maximal ordered leftideal of P.
***Proof*:** Consider that P is not left simple. Then there exists a proper ordered leftideal of P, so $\mho$ is an ordered leftideal of P. We study the adopting two cases.
case 1) $\mho = P$.
By lemma2.4, we have $L(x) \neq P$, $\forall\, x \in P$. In this case, the second condition is fulfilled.
case 2) $\mho \neq P$.

Then $\mho$ is a maximal ordered leftideal of P. Now consider that L is a maximal ordered leftideal of P. Then $L \subseteq \mho \subset P$ because L is a proper ordered leftideal of P. Since L is a maximal ordered leftideal of P, we have $L = \mho$. Hence the unique maximal ordered leftideal of P is $\mho$. By Theorem2.3, we get
(1) $P \setminus \mho = \{x\}$ and $([PPx]] \subseteq \mho$ for some $x \in P$.

(2) $P \setminus \mho \subseteq ([PPx]]$, $\forall\, x \in P \setminus \mho$.

Suppose that $P - \mho = \{x\}$ & $([PPx]] \subseteq \mho$ for some $x \in P$. Then $([PPx]] \subseteq \mho = P \setminus \{x\}$. Since $x \notin \mho$, we have $L(x) = P$. If $x \in ([PPx]]$, then $(x] \subseteq ([PPx]]$. By lemma1.1, we have $P = L(x) = ([PPx] \cup x] = ([PPx]] \cup (x] = ([PPx]] \subseteq ([PP\mho]] \cup \mho = \mho \subseteq P$. Thus $P = \mho$, so it is not possible. Hence $x \notin ([PPx]]$. In this, the third condition is fulfilled.
Now we suppose that $P \setminus \mho \subseteq ([PPx]], \forall\, x \in P \setminus \mho$. To show that $P \setminus \mho = \{y \in P: ([PPy]] = P\}$, assume $y \in P \setminus \mho$. Then $y \in ([PPy]]$, so $(y] \subseteq ([PPy]]$. By lemma1.1, we have $L(y) = ([PPy] \cup y] = ([PPy]] \cup (y] = ([PPy]]$. Since $y \notin \mho$, we have $L(y) = P$. Hence $P = L(y) = ([PPy]]$.

Reversely, assume $y \in P$ be such that $([PPy]] = P$. If $y \in \mho$, then $L(y) \subseteq \mho \subset P$. By lemma1.1, we have $L(y) = ([PPy] \cup y] = ([PPy]] \cup (y] = P \cup (y] = P$. It is not possible, so $y \in P \setminus \mho$. Hence we conclude that $P \setminus \mho = \{y \in P: (PPy] = P\}$. In this,the fourth condition is fulfilled. The theorem proof is hence completed. Theorem2.5 proof is using in the Theorem2.6.

***Theorem2.6*:** One and only one of the adopting four conditions is contented,If P has a zero element and $[PPP] \neq \{0\}$,
(*a*) P is left 0 – simple.
(*b*) $L(x) \neq P$, $\forall\, x \in P$.
(*c*) $\exists\, x \in P \ni L(x) = P, x \notin ([PPx]]$ and $([PPx]] \subseteq \mho = P \setminus \{x\}$ and $\mho$ is the unique maximal ordered leftideal of P.
(*d*) $P \setminus \mho = \{y \in P: ([PPy]] = P\}$ and $\mho$ is the unique maximal ordered leftideal of P.

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