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Certain Properties of Fuzzy Compact Spaces

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ABSTRACT:- *In this paper, we study several properties of fuzzy compactness due to C. L. Chang in fuzzy topological spaces and obtain its several other features.*

Keywords:- *Fuzzy topological space, Fuzzy compactness.*

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations was first introduced by L. A. Zadeh[7] to provide a natural frame work for generalizing many of the concepts of general topology which has useful application in various fields in mathematics. C. L. Chang[2] developed the theory of fuzzy compactness was also studied in this paper. The purpose of this paper is to study this concept in more detail and to obtain several other properties.

1.1. Definition[7] : Let X be a non-empty set and I is the closed unit interval [0, 1]. A fuzzy set in X is a function $u : X \to I$ which assigns to every element $x \in X$. u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X . A member of I^X may also be a called fuzzy subset of X .

1.2. Definition[7]: The union and intersection of fuzzy sets are denoted by the symbols $\cup$ and $\cap$ respectively and defined by

$$\cup u_i = \max \{u_i(x) : i \in J \ and \ x \in X\}$$

$\cap u_i = \min \{u_i(x) : i \in J \ and \ x \in X\}$, where J is an index set .

1.3. Definition[2] : Let u and v be two fuzzy sets in X. Then we define

$(i)\ u = v \ iff\ u(x) = v(x) \ for\ all\ x \in X$

$(ii)\ u \subseteq v \ iff\ u(x) \le v(x) \ for\ all\ x \in X$

$(iii)\ \lambda = u \cup v \ iff\ \lambda(x) = (u \cup v)(x) = \max[u(x), v(x)] \ for\ all\ x \in X$

$(iv)\ \mu = u \cap v \ iff\ \mu(x) = (u \cap v)(x) = \min[u(x), v(x)] \ for\ all\ x \in X$

$(v)\ \gamma = u^c \ iff\ \gamma(x) = 1 - u(x) \ for\ all\ x \in X$.

1.4. Distributive laws[7]: Distributive laws remain valid for fuzzy sets in X i.e. if u , v and w are fuzzy sets in X , then

$(i)\ u \cup (v \cap w) = (u \cup v) \cap (u \cup w)$

$(ii)\ u \cap (v \cup w) = (u \cap v) \cup (u \cap w)$

1.5. Definition[2]: Let X be a non-empty set and $t \subseteq I^X$ i.e. t is a collection of fuzzy set in X. Then T is called a fuzzy topology on X if

$(i)\ 0 , 1 \in t$

$(ii)\ u_i \in t \ for\ each\ i \in J , then \bigcup_i u_i \in t$

$(iii)\ u , v \in t , then\ u \cap v \in t$

The pair (X, t) is called a fuzzy topological space and in short, fts. Every member of t is called a t-open fuzzy set. A fuzzy set is t-closed iff its complements is t-open. In the sequel, when no confusion is likely to arise, we shall call a t-open (t-closed) fuzzy set simply an open (closed) fuzzy set .

1.6. Definition[2]: Let (X,t) and (Y,s) be two fuzzy topological spaces. A mapping $f:(X,t)\to(Y,s)$ is called fuzzy continuous iff the inverse of each s-open fuzzy set is t-open.

1.7. Definition[6]: Let (X,t) be an fts and $A\subseteq X$. Then the collection $t_A=\{u\,|\,A:u\in t\}=\{u\cap A:u\in t\}$ is fuzzy topology on A, called the subspace fuzzy topology on A and the pair (A,t_A) is referred to as a fuzzy subspace of (X,t).

1.8. Definition[2] : A family F of fuzzy sets is a cover of a fuzzy set u iff $u\subseteq\bigcup\{u_i:u_i\in F\}$. It is an open cover iff each member of F is an open fuzzy set. A subcover of F is a subfamily of F which is also a cover .

1.9. Definition[2] : An fts (X,t) is compact iff each open cover has a finite subcover.

1.10. Definition[1]: Let (X,T) be a topological space. A function $f:X\to\mathbf{R}$ (with usual topology) is called lower semi-continuous (l . s . c.) if for each $a\in\mathbf{R}$, the set $f^{-1}(a,\infty)\in T$. For a topology T on a set X , let $\omega(T)$ be the set of all l . s . c. functions from (X,T) to I (with usual topology); thus $\omega(T)=\{u\in I^X:u^{-1}(a,1]\in T,\ a\in I_1\}$. It can be shown that $\omega(T)$ is a fuzzy topology on X.

Let P be a property of topological spaces and FP be its fuzzy topology analogue. Then FP is called a ‘ good extension’ of P “ iff the statement (X,T) has P iff $(X,\omega(T))$ has FP” holds good for every topological space (X,T). Thus characteristic functions are l . s. c.

II. CHARACTERIZATIONS OF FUZZY COMPACTNESS.

Now we obtain some tangible properties of fuzzy compactness.

2.1. Theorem : Let (X,t) be an fts and $A\subseteq X$. If (X,t) is compact and A is closed, then (A,t_A) is compact subspace of (X,t).

Proof : Let $M=\{u_i:i\in J\}$ be an open cover of (A,t_A). Then $A\subseteq\bigcup\{u_i:i\in J\}$. Then by definition of subspace fuzzy topology, there exist $v_i\in t$ such that $u_i=A\cap v_i$. Therefore $A\subseteq\bigcup\{u_i:i\in J\}$ = $\bigcup\{A\cap v_i:i\in J\}$ implies that $A\subseteq A\bigcap[\cup\{v_i:i\in J\}]$. Hence $A\subseteq\bigcup\{v_i:i\in J\}$ and we see that $\{v_i:i\in J\}$ is an open cover of A. Now, $X=A^c\cup A$. Therefore $X=A^c\cup[\cup\{v_i:i\in J\}]$. Since A is closed, it follows that A^c is open. Then the family $H=\{A^c,\{v_i:i\in J\}\}$ is an open cover of X. As (X,t) is compact and A and A^c are disjoint, so we can exclude A^c from this cover. Then there exist $v_{i_1},v_{i_2},\ldots,v_{i_n}\in\{v_i\}$ such that $A\subseteq\bigcup\{v_{i_k}:k=1,2,\ldots\ldots,n\}$. Therefore $A\subseteq A\cap[\bigcup_{k=1}^{n}\{v_{i_k}\}]$. So $A\subseteq\bigcup_{k=1}^{n}\{A\cap v_{i_k}\}$ implies that $A\subseteq\bigcup_{k=1}^{n}\{u_{i_k}\}$. It follows that $\{u_{i_k}:k=1,2,\ldots,n\}$ is a finite subcover of M. Hence (A,t_A) is compact.

2.2. Definition[3]: Let (A,t_A) and (B,s_B) be fuzzy subspaces of fts’s (X,t) and (Y,s) respectively and f is a mapping from (X,t) to (Y,s), then we say that f is a mapping from (A,t_A) to (B,s_B) if $f(A)\subset B$.

2.3. Definition [3]: Let (A,t_A) and (B,s_B) be fuzzy subspaces of fts’s (X,t) and (Y,s) respectively. Then a mapping $f:(A,t_A)\to(B,s_B)$ is relatively fuzzy continuous iff for each $v\in s_B$, the intersection $f^{-1}(v)\cap A\in t_A$.

2.4. Theorem : Let (A,t_A) and (B,s_B) be fuzzy subspaces of fts's (X,t) and (Y,s) respectively with (A,t_A) is compact. Let $f:(A,t_A)\to(B,s_B)$ be relatively fuzzy continuous surjection. Then (B,s_B) is compact.

Proof : Let $v_i \in s_B$ for each $i \in J$ and assume that $\bigcup_{i\in J} v_i = 1$. Then there exist $u_i \in s$ such that $v_i = u_i \cap B$. For each $x \in X$, the inverse $\bigcup_{i\in J} f^{-1}(v_i)(x)$ = $\bigcup_{i\in J} f^{-1}(u_i \cap B)(x)$ = $\bigcup (u_i \cap B)\, f(x)$ = 1. So t_A- open fuzzy sets $f^{-1}(u_i \cap B)$ for each $i \in J$ is cover of A. Since A is compact, then for finitely many indices i_1 , i_2 , ... , i_n $\in J$ such that $\bigcup_{j=1}^{n} f^{-1}(u_{i_j} \cap B) = 1$. Again, let v be any fuzzy set in B, the fact that f is a surjection mapping onto B implies that, for any $y \in B$, we have $f\left(f^{-1}(v)\right)(y) = \sup\{f^{-1}(v)(z) : z \in f^{-1}(y)\}$ = $\sup\{v(f(z)) : f(z) = y\}$ = $\sup\{v(y)\}$ = $v(y)$. Thus $f\left(f^{-1}(v)\right) = v$. Hence 1 = $f(1)$ = $f\left(\bigcup_{j=1}^{n} f^{-1}(u_{i_j} \cap B)\right)$ = $\bigcup_{j=1}^{n} f\left(f^{-1}(u_{i_j} \cap B)\right)$ = $\bigcup_{j=1}^{n} (u_{i_j} \cap B)$. Thus (B,s_B) is compact.

2.5. Definition[4] : An fts $(X\ ,t)$ is said to be fuzzy Hausdorff or fuzzy - T_2 iff for all $x\ ,\ y \in X$, $x \neq y$, there exist $u\ ,\ v \in t$ such that $u(x) = 1$, $v(y) = 1$ and $u \subseteq 1 - v$.

2.6. Theorem : Let $(X\ ,t)$ be a fuzzy Hausdorff space and A be a compact subset of $(X\ ,t)$. Suppose $x \in A^c$, then there exist $u\ ,\ v \in t$ such that $u(x) = 1$, $A \subseteq v^{-1}(0\ ,1]$ and $u \subseteq 1 - v$.

Proof : Let $y \in A$. Since $x \notin A$ $(x \in A^c)$, then clearly $x \neq y$. As $(X\ ,t)$ is fuzzy Hausdorff , then there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \subseteq 1 - v_y$. Hence $A \subseteq \bigcup \{v_y : y \in A\}$ i.e. $\{v_y : y \in A\}$ is an open cover of A. Since A is compact, then there exist v_{y_1} , v_{y_2} , ... , $v_{y_n} \in \{v_y\}$ such that $A \subseteq v_{y_1} \cup v_{y_2} \cup .... \cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup .... \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap .... \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $v\ ,\ u \in t$. Furthermore, $A \subseteq v^{-1}(0\ ,1]$ and $u(x) = 1$, since each $u_{y_i}(x) = 1$ for each i.

Finally, we have to show that $u \subseteq 1 - v$. As $u_y \subseteq 1 - v_y$ implies $u \subseteq 1 - v_y$. Since $u_{y_i}(x) \leq 1 - v_{y_i}(x)$ for all i, then $u \subseteq 1 - v$. If not, there exists $x \in X$ such that $u_y(x) \leq 1 - v(x)$. We have $u_y(x) \leq u_{y_i}(x)$ for all i. Then for some i, $u_{y_i}(x) \leq 1 - v_{y_i}(x)$. But this is a contradiction as $u_{y_i}(x) \leq 1 - v_{y_i}(x)$ for all i. Hence $u \subseteq 1 - v$.

2.7. Theorem : Let $(X\ ,t)$ be a fuzzy Hausdorff space and A , B be disjoint compact subsets of $(X\ ,t)$. Then there exist $u\ ,\ v \in t$ such that $A \subseteq u^{-1}(0\ ,1]$, $B \subseteq v^{-1}(0\ ,1]$ and $u \subseteq 1 - v$.

Proof : Let $y \in A$. Then $y \notin B$, as A and B are disjoint. Since B is compact, then by (2.7), there exist u_y , $v_y \in t$ such that $u_y(y) = 1$, $B \subseteq v_y^{-1}(0\ ,1]$ and $u_y \subseteq 1 - v_y$. Since $y \in u_y$, then $\{u_y : y \in A\}$ is an open cover of A. As A is compact , then there exist u_{y_1} , u_{y_2} , ... , $u_{y_n} \in \{u_y\}$ such that

$A \subseteq u_{y_1} \cup u_{y_2} \cup ...... \cup u_{y_n}$. Furthermore, $B \subseteq v_{y_1} \cap v_{y_2} \cap .... \cap v_{y_n}$, as $B \subseteq v_{y_i}$ for each i. Now, let $u = u_{y_1} \cup u_{y_2} \cup .... \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap .... \cap v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0\,,1]$ and $B \subseteq v^{-1}(0\,,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $u\,,\, v \in t$.

Finally, we have to show that $u \subseteq 1 - v$. First, we observe that $u_{y_i} \subseteq 1 - v_{y_i}$ for each i, implies that $u_{y_i} \subseteq 1 - v$ for each i and it is clear that $u \subseteq 1 - v$.

2.8. Theorem : Let A be a compact subset of a fuzzy Hausdorff space (X, t). Then A is closed.

Proof : Let $x \in A^c$. We have to show that there exists $u \in t$ such that u(x) = 1 and $u \subseteq A^p$, where A^p is the characteristic function of A^c. Now, let $y \in A$, then there exist $u_y\,,\, v_y \in t$ such that $u_y(x) = 1,\ v_y(y) = 1$ and $u_y \subseteq 1 - v_y$. Thus we see that $A \subseteq \bigcup\{v_y : y \in A\}$ i.e. $\{v_y : y \in A\}$ is an open cover of A. Since A is compact, so it has a finite subcover, say $v_{y_1}, v_{y_2}, ..., v_{y_n} \in \{v_y\}$ such that $A \subseteq v_{y_1} \cup v_{y_2} \cup .... \cup v_{y_n}$. Again, let $u = u_{y_1} \cap u_{y_2} \cap .... \cap u_{y_n}$. Hence we observe that $u(x) = 1$ and $u \cap (v_{y_1} \cup v_{y_2} \cup .... \cup v_{y_n}) = 0$. For each $z \in A$, it is clear that $\bigcup\{v_{y_k}\}(z) = 1\ (1 \le k \le n)$. Thus $u(z) = 0$ and hence $u \subseteq A^p$. Therefore, A^c is open and so A is closed.

2.9. Definition[1] : An fts $(X\,, t)$ is said to be fuzzy regular iff for each $x \in X$ and $u \in t^c$ with $u(x) = 0$, there exist $v\,,\, w \in t$ such that $v(x) = 1,\ u \subseteq w$ and $v \subseteq 1 - w$.

2.10. Theorem : Let $(X\,, t)$ be a fuzzy regular space and A be a compact subset of $(X\,, t)$. Suppose $x \in A$ and $u \in t^c$ with $u(x) = 0$. Then there exist $v\,,\, w \in t$ such that $v(x) = 1,\ u \subseteq w,\ A \subseteq v^{-1}(0\,,1]$ and $v \subseteq 1 - w$.

Proof : Suppose $x \in A$ and $u \in t^c$ we have $u(x) = 0$. Since $(X\,, t)$ is fuzzy regular, then there exist $v_x\,,\, w_x \in t$ such that $v_x(x) = 1$, $u_x \subseteq w_x$ and $v_x \subseteq 1 - w_x$. Hence $A \subseteq \bigcup\{v_x : x \in A\}$ i.e. $\{v_x : x \in A\}$ is an open cover of A. Since A is compact, so it has a finite subcover, say $v_{x_1}, v_{x_2}, ..., v_{x_n} \in \{v_x\}$ such that $A \subseteq v_{x_1} \cup v_{x_2} \cup .... \cup v_{x_n}$. Now, let $v = v_{x_1} \cup v_{x_2} \cup .... \cup v_{x_n}$ and $w = w_{x_1} \cap w_{x_2} \cap .... \cap w_{x_n}$. Thus we see that v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $v\,,\, w \in t$. Furthermore, $u \subseteq w,\ A \subseteq v^{-1}(0\,,1]$ and $v(x) = 1$.

Finally, we have to show that $v \subseteq 1 - w$. As $v_{x_i} \subseteq 1 - w_{x_i}$ for each i implies that $v_{x_i} \subseteq 1 - w$ for each i and it is clear that $v \subseteq 1 - w$.

2.11. Theorem : Let $(X\,, t)$ be a fuzzy regular space and A , B be disjoint compact subsets of $(X\,, t)$. For each $x \in X$ and $u \in t^c$ with $u(x) = 0$, there exist $v\,,\, w \in t$ such that $v(x) = 1$, $A \subseteq v^{-1}(0\,,1]$, $B \subseteq w^{-1}(0\,,1]$ and $v \subseteq 1 - w$.

Proof : Suppose for each $x \in X$ and $u \in t^c$ we have $u(x) = 0$. Let $x \in A$. Then $x \notin B$, as A and B are disjoint. Since B is compact then by (2.11) , there exist $v_x\,,\, w_x \in t$ such that $v_x(x) = 1$, $B \subseteq w_x^{-1}(0\,,1]$ and $v_x \subseteq 1 - w_x$. Since $v_x(x) = 1$, then $\{v_x : x \in A\}$ is an open cover of A. As A is compact, so it has a finite subcover, say $v_{x_1}, v_{x_2}, ..., v_{x_n} \in \{v_x\}$ such that

$A \subseteq v_{x_1} \cup v_{x_2} \cup .... \cup v_{x_n}$. Furthermore, $B \subseteq w_{x_1} \cap w_{x_2} \cap .... \cap w_{x_n}$, as $B \subseteq w_{x_i}$ for each i. Now, let $v = v_{x_1} \cup v_{x_2} \cup .... \cup v_{x_n}$ and $w = w_{x_1} \cap w_{x_2} \cap .... \cap w_{x_n}$. Thus we see that $A \subseteq v^{-1}(0\ ,1]$, $B \subseteq w^{-1}(0\ ,1]$. Hence v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $v\ ,\ w \in t$.

Finally, we have to show that $v \subseteq 1 - w$. As $v_{x_i} \subseteq 1 - w_{x_i}$ for each i, we have $v_{x_i} \subseteq 1 - w$ for each i
and it is then clear that $v \subseteq 1 - w$.

2.12. Theorem : A topological space $(X\ ,T)$ is compact iff $(X\ ,\omega(T))$ is fuzzy compact.

Proof : Suppose $(X\ ,T)$ is compact. Let $\{u_i : i \in J\}$ be an open cover of $(X\ ,\omega(T))$ i.e $X = \bigcup_{i \in J} \{u_i : u_i \in \omega(T)\}$. Then $u_i^{-1}(a\ ,1] \in T$ and $\{u_i^{-1}(a\ ,1] : u_i^{-1}(a\ ,1] \in T\}$ is an open cover of $(X\ ,T)$. Since $(X\ ,T)$ is compact, so it has a finite subcover, say $u_{i_k}^{-1}(a\ ,1] \in T \quad (1 \le k \le n)$ such that $X = u_{i_1}^{-1}(a\ ,1] \cup u_{i_2}^{-1}(a\ ,1] \cup .... \cup u_{i_n}^{-1}(a\ ,1]$. Now, we can write $X = u_{i_1} \cup u_{i_2} \cup .... \cup u_{i_n}$ and it is seen that $\{u_{i_k}\}\ (1 \le k \le n)$ is a finite subcover of $\{u_i : i \in J\}$. Thus $(X\ ,\omega(T))$ is fuzzy compact.

Conversely, suppose that $(X\ ,\omega(T))$ is fuzzy compact. Let $\{V_j : j \in J\}$ be an open cover of $(X\ ,T)$ i.e. $X = \bigcup_{j \in J} \{V_j : j \in J\}$. Since 1_{V_j} are l . s. c. then $1_{V_j} \in \omega(T)$ and $\{1_{V_j} : 1_{V_j} \in \omega(T)\}$ is an open cover of $(X\ ,\omega(T))$. Since $(X\ ,\omega(T))$ is fuzzy compact, so it has a finite subcover, say $\{1_{V_{jk}} : 1_{V_{jk}} \in \omega(T)\} \quad (1 \le k \le n)$ such that $X = 1_{V_{j1}} \cup 1_{V_{j2}} \cup .... \cup 1_{V_{jn}}$. Now, we can write $X = V_{j_1} \cup V_{j_2} \cup .... \cup V_{j_n}$ and it is seen that $\{V_{j_k}\}\ (1 \le k \le n)$ is a finite subcover of $\{V_j : j \in J\}$. Thus $(X\ ,T)$ is compact.

We hope to study certain other properties of this concept in our next work.

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