International Refereed Journal of Engineering and Science (IRJES)
ISSN (Online) 2319-183X, (Print) 2319-1821
Volume 2, Issue 6(June 2013), PP.33-43
www.irjes.com

A Critical Review of Some Properties and Applications of the Negative Binomial Distribution (NBD) and Its Relation to Other Probability Distributions.

Mr INUWA, Paul Dalatu
Department of Mathematical Sciences Faculty of Science Adamawa State University, Mubi

ABSTRACT:- *Binomial experiment involves event of two outcomes considered as either success or failure, each trial is independent of each other and the probability of a success remain the same for each trial. This give rise to a special probability distribution called binomial distribution. Negative binomial distribution (NBD) being the use of binomial theorem with a negative exponent has been playing an important role in areas of health, demographic, communications, climatology, and engineering and like. It is in the light of these, that we attempt to make a critical review of some properties, applications and its relations to other probability distributions with the belief of its usefulness to statisticians.*

Keywords:- *Negative binomial, Logarithmic series distribution, Geometric distribution, Poisson distribution, Gamma-Poisson mixture.*

I. INTRODUCTION

According to Cook in 2008, the name comes from applying the general form of the binomial theorem with a negative exponent.

$$1 = p^r p^{-r} = p^r (1-q)^{-r} = p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-q)^x$$

The xth term in the series above is

$$\binom{-r}{x} p^r (-q)^x = (-1)^x \binom{-r}{x} p^r q^x = \binom{r+x-1}{x} p^r (1-p)^x$$

Which is the probability that $X = X$ where X ~negative binomial with parameters r and p.

It can be seen according to Wikipedia in 2012 that, the probability mass function of the negative binomial distribution is given by

$$f(k) \equiv \Pr(X = k) = \binom{k+-1}{k} (1-p)^r p^k \quad , \quad \text{for} \quad k = 0,1,2,\ldots$$

Here the quantity in parentheses is the binomial coefficient, and is equal to

$$\binom{k+r-1}{k} = \frac{(k+r-1)!}{k!(r-1)!} = \frac{(k+r-1)(k+r-2)\ldots(r)}{k!}.$$

This quantity can alternatively be written in the following manner, explaining the name "negative binomial":

$$\frac{(k+r-1)\ldots(r)}{k!} = (-r)^k \frac{(-r)(-r-1)(-r-2)\ldots(-r-k+1)}{k!} = (-1)^k \binom{-r}{k}. \quad \ldots \quad (*)$$

To understand the above definition of the probability mass function, note that the probability for every specific sequence of k successes and r failures is $(1-p)^r p^k$, because the outcomes of the $k + r$ trials are supposed to happen independently. Since the r^{th} failure comes last, it remains to choose the k trials with successes out of the remaining $k|+r-1$ trials. The above binomial coefficient, due to its combinatorial interpretation, gives precisely the number of all these sequences of length $k|+r-1$.

Extension To Real-Valued R

It is possible to extend the definition of the negative binomial distribution to the case of a positive real parameter r. Although, it is impossible to visualize a non-integer number of "failures", we can still formally define the distribution through its probability mass function.

As before, we say that X has a negative binomial distribution if it has a probability mass function:

$$f(k) \equiv \Pr(X = k) = \binom{k+r-1}{k}(1-p)^r p^k \quad \text{, for } k = 0,1,2,\ldots$$

Here r is a real and positive number. The binomial coefficient is then defined by the multiplicative formula and can also be rewritten using the gamma function:

$$\binom{k+r-1}{k} = \frac{(k+r-1)(k+r-2)\ldots(r)}{k!} = \frac{\Gamma(k+r)}{k!\Gamma(r)}.$$

Note that by the binomial series and $(*)$ above, for every $0 \le p < 1$,

$$(1-p)^{-r} = \sum_{k=0}^{\infty}\binom{-r}{k}(-p)^k = \sum_{k=0}^{\infty}\binom{k+r-1}{k}p^k,$$

hence the terms of the probability mass function indeed add up to one.

Pascal and Fermat in 1865, discussed negative binomial distribution extensively but Montmort in 1714, was one of the earliest contributors. The NBD is unimodular and positively skewed. The NBD has many forms.

1.0.1 The Anscombe (1950) form.

There are many forms of the NBD throughout the literature. The NBD in Anscombe form is of the form:

$$P(j) = \frac{\Gamma(k+j)}{j!\Gamma(k)}(1+m/k)^{-k}\left(\frac{m}{m+k}\right)^j, \; j = 0,1,2,\ldots; m > 0. \quad \ldots$$

(1.1)

This form depends on two parameters, m (mean) and the exponent, k; which is positive but usually non-integral.

1.0.2 The Pascal form.

The Pascal forms are often expressed as:

$$P(j) = \binom{j+r-1}{j}p^r q^j; \; j = 0,1,2,\ldots; r = 1,2,3,\ldots; p+q = 1. \quad \ldots$$

(1.2)

$$P(j) = \binom{n-1}{r-1}p^r q^j; \; j = 0,1,2,\ldots; p+q = 1; r = 1,2,3,\ldots; n = r+j. \quad (1.3) \quad \ldots$$

The NBD is a general class of distributions often referred to as contagious distributions. These forms of distributions are treated extensively by Beall (1940), Feller (1943), Rutherford (1954) and Douglas (1955).

II. OCCURRENCE

NBD can arise in a number of ways:

2.1 Inverse Binomial Sampling (IBS).

IBS is of primary importance in the study of the NBD. It is applied in the study of population models where we obtain constant rates of birth and death, and a constant rate of immigration can also be modeled with a NBD. If a proportion p of individuals in a population possesses a certain characteristic, the number of observations in excess of r which must be taken to obtain exactly r individuals with that characteristic can be expressed in the Pascal form of the NBD (Yule, 1910, and Haldane, 1945).

2.2 Compound Poisson Model.

Suppose that X is a random variable with a Poisson distribution with mean θ. The probability density function (pdf) is given by the relation

$$f(\theta) = \theta^{k-1}\exp\left(\frac{-\theta k}{m}\right)\{(m/k)^k\Gamma(k)\}; \qquad \theta > 0.$$

. . . (2.1)

Where M and k are given parameters; $m, k > 0$, consequently the probability of the random variable X is given by

$$P(X=j)=\begin{cases} \int_0^\infty \frac{\theta^j e^{-\theta}}{j!} f(\theta)d\theta \\ \frac{\Gamma(k+j)}{j!\Gamma(k)}(1+m/k)^{-k}(m/(m+k))^j;\ j=0,1,2\ldots \end{cases}$$

. . . (2.2)

It is the model developed by Greenwood and Yule in 1920 and is the Anscombe form of the NBD.

This model has been widely applied in accident statistics. The works of Greenwood and Yule in 1920, Adelstein in 1952, and Irwin in 1964, dwelt extensively on this model. Under certain assumptions, this model had also been applied to the study of consumer purchasing behavior. Please refer to (Ehrenberg, 1959; Chatfield et al, 1966; Chatfield and Goodhart, 1970) where were excellent literature on this model were found. In arriving at their modeling certain assumptions were made, primary among which are:

2.2.1 Randomly Distributed Colonies (RDC).

RDC is a mathematical model which leads to the negative binomial distribution (Jones, etal, 1984 and Anscombe, 1949), in their study of the number of bacterial counts in the soil concluded that the distribution is negative binomial.

2.2.2 Immigration-Emigration-Birth-Death process.

Some simple model of population growth in which there are constant IEBD leads to a negative binomial distribution for the population size. (Mckendrick, 1914; Yule, 1924; Furry, 1939; and Kendall, 1949). The model has been applied to the spread of an infectious disease in a community (Irwin, 1954).

III. PROPERTIES OF THE NBD.

We discuss briefly the properties of the NBD in this section. Consider the Pascal form:

$$P(j)=\binom{j+r-1}{j}p^r q^j;\ j=0,1,2,\ldots \quad \ldots$$

(3.1)

For $j=0,\ p(0)=p^r$.

Using the recurrence relationship:

$$P(j)=q(1+(r-1)/j)p(j-1);\ j=1,2,3,\ldots \quad \ldots$$

(3.2)

We can calculate these probabilities.

The moment generating function (mgf) of this distribution is given by

$$M_X(t)=\sum_{j=0}^{\infty}\left[q(1+(r-1)/j)p^r q^j e^{tj}\right]$$
$$=p^r\left(1-qe^t\right)^{-r}. \quad \ldots$$

(3.3)

Furthermore, it can be shown

(i) $E[X]=rq/p,\quad Var(X)=rq/p^2$

(ii) The sum of any finite number of independent negative binomial variates with the same p is also a negative binomial variate. If $Var(X) > E[X]$, w e said to have over dispersion.

Pearson and Fieller (1933) pointed out that the Pascal form of the NBD's cumulative density function (c.d.f) is given as

$$F(X)=\sum_{j=0}^{X}\binom{j+r-1}{j}p^r q^j$$
$$=I_P(r, X+1) \quad \ldots$$

(3.4)

And can be computed by using tables of the incomplete Beta function. This was rediscovered by Patil in 1960.

IV. DISTRIBUTIONS WHICH CAN BE USED TO APPROXIMATE THE NBD.

We shall consider three distributions that can be used to approximate the NBD.

4.1 The Poisson distribution.

We know that the moment generating function of:

$$P(j)=\binom{j+r-1}{j}p^r q^j;\ j=0,1,2\ldots$$

is given by

$$M_X(t)=p^r\left(1-qe^t\right)^{-r}.$$

If $r\to\infty$ and $q\to 0, qr\to m$ say, $m>0$, then

$M_X(t)\to e^{m\left(e^t-1\right)}$ Of the mgf of the Poisson distribution with mean, m.

It follows that if r is large and q is small, then

$$\binom{j+r-1}{j}p^r q^j \doteq (qr)^j e^{-qr}/j!,\ j=0,1,2\ldots\quad .$$

4.2 The Normal Distribution.

In (3.1), if X has a NBD, its mean and variance given by rq/p and rq/p^2 .

By the central limit theorem for r large, the distribution of X can be approximated by that of the normal distribution with mean and variance given as rq/p and rq/p^2 respectively.

4.3 Logarithmic series distribution.

The Anscombe form:

$$P(j)=\frac{\Gamma(k+j)}{j!\Gamma(k)}(1+m/k)^{-k}\left(m/(m+k)\right)^j.$$

Fisher, et al, in 1943 approximated this to

$$P(X)=-q^X/\{X\ln(1-q)\};\ X=1,2,3,\ldots;\qquad 0<q<1,$$

which is referred to as the logarithmic series distribution.

V. ESTIMATION OF PARAMETERS.

We can estimate the parameters in the NBD. The works of Haldane in 1943, Finney in 1949, Fisher in 1941, and Anscombe in 1950; discussed ways of estimating the parameters of the NBD.

VI. THE RELATIONS

According to Cook in 2008, the name comes from applying the general form of the binomial theorem with a negative exponent. $1-p^r p^{-r}=p^r(1-q)^{-r}=p^r\sum_{x=0}^{\infty}\binom{-r}{x}(-q)^x$.

The xth term in the series above is

$$\binom{-r}{x}p^r(-q)^x=(-1)^x\binom{-r}{x}p^r q^x=\binom{r+x-1}{x}p^r(1-p)^x.$$

Which is the probability that $X=x$ where $X\sim$ negative binomial with parameters r and p.

Sadinle in 2008 explained that, though probability distributions model of real situations, their mathematical expressions are sometimes connected. In his paper, he stated that integrals and derivatives of the series associated to the negative binomial and logarithmic series are used to find the relationship between these distributions. That the parameter of the number of failures of the negative binomial distribution is found to be the number of derivatives of the logarithmic series needed to find the negative binomial series. He further said the relationship between the logarithmic series distribution and the negative binomial is not as explicit as the relation between the geometric distribution and the negative binomial distribution.

The compound negative binomial distribution with exponential claim amounts (severity) distribution as shown to be equivalent to a compound binomial distribution with exponential claim amounts (severity) with a parameter (Panjer and Willmot, 1981).

Although, Ghitany et al in 2002, clarified some issues that, a certain mixture distribution arises when all (or some) parameters of a distribution vary according to some probability distribution, called the *mixing distribution*.

A well-known example of discrete-type mixture distribution is the negative binomial distribution which can be obtained as a Poisson mixture with gamma mixing distribution.

6.1 Relation to other distributions.

In a sequence of independent Bernoulli (p) trails, let the random variable X be denote the trial at which the rth success occurs, where r is a fixed integer, then

$$P(X = X \setminus r, p) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \, ; x = r, r+1, \ldots, \quad . \quad . \quad . \quad (6.1)$$

And we say that X has a negative binomial (r, p) distribution.

The negative binomial distribution is sometimes defined in terms of the random variable Y= number of failure before rth success. This formulation is statistically equivalent to the one given above in terms of X = trial at which the rth success occurs, since Y = X- r. The alternative form of the negative binomial distribution is

$$P(Y = y) = \binom{r+y-1}{y} p^r (1-p)^y, \; y = 0,1,2,\ldots.$$

The negative binomial distribution gets its name from the relationship

$$\binom{r+y-1}{y} = (-1)^y \binom{-r}{y} = (-1) \frac{y(-r)(-r-1)\ldots(-r-y+1)}{(y)(y-1)\ldots(2)(1)} \quad . \quad . \quad . \quad (6.2)$$

This is the defining equation for binomial coefficient with negative integers. Along with (6.2), we have $\sum_y P(Y = y) = 1$, from the negative binomial expansion which states that:

$$(1+t)^{-r} = \sum_k \binom{-r}{k} t^k$$

$$= \sum_k (-1)^k \binom{r+k-1}{k} t^k$$

$$EY = \sum_{y=0}^{\infty} y \binom{r+y-1}{y} p^r (1-p)^y$$

$$= \sum_{y=1}^{\infty} \frac{(r+y-1)!}{(y-1)!(r-1)!} p^r (1-p) y$$

$$= \frac{r(1-p)}{p} \sum_{z=0}^{\infty} \binom{r+1+z-1}{z} p^{r+1} (1-p)^z$$

$$= r \frac{1-p}{p}.$$

A similar calculation showed that,

$$VarY = \frac{r(1-p)}{p^2}.$$

6.1.1 Geometric distribution

The geometric distribution is the simplest of the waiting time distributions and is a special case of the negative binomial distribution. Let $r = 1$ in (6.1) we have;

$$P(X = x / p) = p(1-p)^{x-1}; x = 1,2,3,\ldots,$$

which defines the pmf of a geometric random variable X with success probability p. X can be interpreted as the trial at which the first success occurs, so we are ‘’ waiting for a success’’. The mean and variance of X can be calculated by using the negative binomial formulas and by writing X= Y+1 to obtain;

$$EX = EY + 1 = \frac{1}{P}, and VarX = \frac{1-p}{p^2}.$$

The geometric distribution has an interesting property, known as the ‘’memory less’’ property. For integers $s > t$, it is the case that;

$$P(X > s : X > t) = p(X > s - t), \quad \ldots$$

(6.3)

That is, the geometric distribution ‘’ forgets ‘’ what has occurred. The probability of an additional $s - t$ failures, having already observed t failures, is the same as probability of observing $s - t$ failures at the start of the sequence. To establish (6.3), we first note that for any integer n;

$$p(X > n) = p(no\ success\ in\ n\ trials) = (1-p)^n$$ and hence

$$p(X > s : X > t) = \frac{p(X > s\ and\ X > t)}{p(X > t)} = \frac{p(X > s)}{p(X > t)}$$

$$(1 - p^{s-t}) = p(X > s - t).$$

6.1.2 Binomial-Exponential and Negative Binomial Models.

Given by Panjer and Willmot in 1981, they considered the distribution of;

$$S = X_1 + X_2 + X_3 + \ldots + X_N \quad \ldots$$

(6.4)

Where $X_1 + X_2 + X_3 \ldots$ are independently and identically distributed random variables with common exponential distribution function;

$$F_X(x) = 1 - e^{-\lambda x}, x \geq 0, \quad \ldots$$

(6.5)

And N is an integer valued random variable with probability function

$$P_n = p_r\{N = n\}, n = 0,1,2,\ldots \quad \ldots$$

(6.6)

Then the distribution function of S is given by

$$F_S(x) = \sum_{n=0}^{\infty} p_n F_X^n(x), x > 0 \quad \ldots$$

(6.7)

If $M_X(t), M_N(t) and M_S(t)$ are the associated moment generating functions, then

$$M_S(t) = E_N E_X\left[e^t(X_1 + \ldots + X_N)/N = n\right] \quad \ldots$$

(6.8)

$$= \sum_{n=0}^{\infty} p_n \{M_X(t)\}^n$$

$$= M_N(l_n M_X(t))$$

The moment generating function of the exponential distribution (6.5) is;

$$M_X(t) = \frac{\lambda}{\lambda - t} \quad \ldots$$

(6.9)

First, consider the binomial distribution with probability function

$$P_n = \binom{n}{m} q^n p^{m-n}, \quad \ldots$$

(6.10)

And moment generating function;

$$M_N(t)=\left(p+qe^t\right)^m \quad \ldots$$

(6.11)

Where $p+q=1$. Then, for the compound binomial distribution with exponential claim amounts (severity), (6.8) becomes

$$M_S(t)=\left(p+q\frac{\lambda}{\lambda-t}\right)^m$$

$$=\left(\frac{\lambda-pt}{\lambda-t}\right)^m \quad \ldots$$

(6.12)

Now consider the negative binomial with probability function;

$$P_n=\binom{\alpha+n-1}{n}p^\alpha q^n \quad \ldots$$

(6.13)

And the moment generating function is;

$$M_N(t)=\left(\frac{p}{1-qe^t}\right)^\alpha \quad \ldots$$

(6.14)

Where $p+q=1$. Then, for the compound negative binomial with exponential claim amounts (severity), (6.8) becomes;

$$M_S(t)=\left(\frac{p}{1-q\dfrac{\lambda}{\lambda-t}}\right)^\alpha \quad \ldots$$

(6.15)

$$\left(\frac{p\lambda-pt}{p\lambda-t}\right)^\alpha .$$

Comparing (6.12) and (6.15), one note that they are of identical form provided that α is integer valued. Hence, the Negative Binomial-Exponential model is equivalent to a Binomial-Exponential model.

6.1.3 Negative binomial can be approximated by Poisson with mean λ.

We will relate how negative binomial distribution can be approximated by Poisson with mean λ. Based on the works of Zelterman in 2004, related the distributions as; suppose we continue to sample Bernoulli distributed events with probability parameter P until we obtain c successes or 1`s. The value of c = 1, 2, . . is determined before the sampling begins. The sampling process ends with the observation of the $c-th$ success. The negative binomial distribution describes the number of failures (0`s) observed before the cth success has been achieved.

The number of failures Y until the c^{th} success follows the negative binomial distribution with mass function

$$P_r[Y=y]=\binom{c+y-1}{c-1}p^c(1-p)^y,\ for\ \ y=0,1,\ldots \quad \ldots$$

(6.16)

The name 'negative binomial 'and the proof that the mass function (6.16) sums to one comes from the following expansion. For P near zero, we have

$$(Q-P)^{-c}=Q^{-c}+\frac{c}{1!}PQ^{-c-1}+\frac{c(c+1)}{2!}P^2Q^{-c-2}+\ldots$$

Then set $Q=P+1$ and we write

$$Q^{-c}=[(Q-P)/Q]^c=(1-P/Q)^c \quad ,$$

to show

.

$$(Q-P)^{-c}=(1-P/Q)^{c}+\frac{c}{1!}(P/Q)(1-P/Q)^{c}+\frac{c(c+1)}{2!}(P/Q)^{2}(1-P/Q)^{c}+\ldots$$

These terms are the same as those in (1.16), where

$$P=(1-p)/p.$$

The expected value of the negative binomial random variable with mass function in (1.16) is

$$E[Y]=\mu=c(1-p)/p \qquad . \; . \; . \quad (6.17)$$

and the variance is

$$Var[Y]=c(1-p)/p^{2}.$$

The variance of the negative binomial distribution is always larger than its mean.
The probability generating function is

$$G(t)=E\left[t^{Y}\right]=\left[(1-p)/(1-pt)\right]^{c}$$

The factorial moment generating function is

$$G(1+t)=E\left[(1+t)^{Y}\right]=\left[1-(1-p)t/p\right]^{-c}$$

and the factorial moments of the negative binomial distribution are

$$E\left[Y^{(k)}\right]=(c+k-1)^{(k)}\left[(1-p/p)\right]^{k} \quad for \;\; k=0,1,\ldots \qquad .$$

If the c parameter is large and p approaches one in such a manner that the mean $c(1-p)/p$ approaches a finite non zero limit λ, then the negative distribution can be approximated by the Poisson with mean λ.
For further reading and detailed plotted figures on the negative binomial distribution, with various values of c and p;(Zelterman, 2004).

6.1.4 Poisson distribution.

Consider a sequence of negative binomial distributions where the stopping parameter r goes to infinity, whereas the probability of success in each trial, p, goes to zero in such a way as to keep the mean of the distribution constant. Denoting this mean λ, the parameter p will have to be

$$\lambda=r\frac{p}{1-p}\Rightarrow p=\frac{\lambda}{r+\lambda}.$$

Under this parameterization the probability mass function will be

$$f(k)=\frac{\Gamma(k+r)}{k!\,\Gamma(r)}(1-p)^{r}p^{k}=\frac{\lambda^{k}}{k!}\cdot\frac{\Gamma(r+k)}{\Gamma(r)(r+\lambda)^{k}}\cdot\frac{1}{\left(1+\frac{\lambda}{r}\right)^{r}}.$$

. . (6.18)

Now if we consider the limit as $r\to\infty$, the second factor will converge to one, and the third to the exponent function:

$\lim_{r\to\infty} f(k)=\frac{\lambda^{k}}{k!}.1.\frac{1}{e^{\lambda}}$, which is the mass function of a Poisson-distributed random variable with expected value λ.

In other words, the alternatively parameterized negative binomial distribution converges to the Poisson distribution and r controls the deviation from the Poisson. This makes the negative binomial distribution suitable as a robust alternative to the Poisson, which approaches the Poisson for larger r, but which has larger variance than Poisson for small r.

$$\text{Poisson}(\lambda)=\lim_{r\to\infty} NB\left(r,\frac{\lambda}{\lambda+r}\right).$$

Gamma-Poisson mixture 6.1.5

The negative binomial distribution also arises as a continuous mixture of Poisson distributions (i.e. a compound probability distribution) where the mixing distribution of the Poisson rate is a gamma distribution. That is, we can view the negative binomial as a Poisson λ distribution, where λ is itself a random variable, distributed according to $Gamma\left(r, p/(1-p)\right)$.

Formally, this means that the mass function of the negative binomial distribution can be written as

$$f(k)=\int_0^\infty f_{Poisson(\lambda)}(k).f_{Gamma\left(r,\frac{p}{1-p}\right)}(\lambda)d\lambda$$

$$=\int_0^\infty \frac{\lambda^k}{k!}e^{-\lambda}.\lambda^{r-1}\frac{e^{-\lambda(1-p)/p}}{\left(\frac{p}{1-p}\right)^r\Gamma(r)}d\lambda$$

$$=\frac{(1-p)^r p^{-r}}{k!\Gamma(r)}\int_0^\infty \lambda^{r+k-e}e^{-\lambda/p}d\lambda$$

$$=\frac{(1-p)^r p^{-r}}{k!\Gamma(r)}p^{r+k}\Gamma(r+k)$$

$$=\frac{\Gamma(r+k)}{k!\Gamma(r)}(1-p)^r p^k. \qquad ... \quad (6.19)$$

Because of this, the negative binomial distribution is also known as the *gamma-Poisson (mixture) distribution*.

6.1.6 Representation as compound Poisson distribution

The Poisson distribution can also be seen from Wikipedia, 2012, that the negative binomial distribution $NB(r, p)$ can be represented as a compound Poisson distribution: Let $\{Y_n, n\in N_0\}$ denote a sequence of independent and identically distributed random variables, each one having the logarithmic distribution $\log(p)$, with probability mass function

$$f(k)=\frac{-p^k}{k\ln(1-p)}, \quad k\in\mathbb{N}.$$

Let N be a random variable, independent of the sequence, and suppose that N has a Poisson distribution with mean $\lambda=-r\ln(1-p)$. Then the random sum

$$N=\sum_{n=1}^{N}Y_n$$

is $NB(r, p)-$ distributed. To prove this, we calculate the probability generating function G_X of X, which is the composition of the probability generating functions G_N and G_{Y1}. Using

$$G_N(z)=\exp(\lambda(z-1)), \qquad z\in\mathbb{R}, \text{ and}$$

$$G_{Y1}(z)=\frac{\ln(1-pz)}{\ln(1-p)}, \qquad |z|< \frac{1}{p},$$

we obtain

$$G_X(z)=G_N(G_{Y1}(z)) \qquad ...$$

(6.20)

$$=\exp\left(\lambda\left(\frac{\ln(1-pz)}{\ln(1-p)}-1\right)\right)$$

$$= \exp\left(-r\left(\ln(1-pz) - \ln(1-p)\right)\right)$$
$$= \left(\frac{1-p}{1-pz}\right)^r, \qquad |z| < \frac{1}{p},$$

Which is the probability generating function of the $NB(r, p)$ distribution.

6.1.7 Sum of geometric distributions.

If Y_r is a random variable following the negative binomial distribution with parameters r and p, and support $\{0,1,2,...\}$,

then Y_r is a sum of r independent variables following the geometric distribution $(on\{0,1,2,3,...\})$ with parameter $1-p$.

As a result of the central limit theorem, Y_r (properly scaled and shifted) is therefore approximately normal for sufficiently large r.

Furthermore, if B_{s+r} is a random variable following the binomial distribution with parameters $s+r$ and $1-p$, then

$$\Pr(Y_r \le s) = 1 - I_p(s+1, r)$$
$$= 1 - I_p((s+r)-(r-1), (r-1)+1)$$
$$= 1 - \Pr(B_{s+r} \le r-1)$$
$$= \Pr(B_{s+r} \ge r) \qquad \ldots$$

(6.21)

$= \Pr$(after $s + r$ trials, there are at least r successes).

In this sense, the negative binomial distribution is the "inverse" of the binomial distribution. The sum of independent negative-binomially distributed random variables r_1 and r_2 with the same value for parameter p is negative-binomially distributed with the same p but with "$r - value$" $r_1 + r_2$.

The negative binomial distribution is infinitely divisible, i.e., if Y has a negative binomial distribution, then for any positive integer n, there exist independent identically distributed random variables $Y_1,..., Y_n$ whose sum has the same distribution that Y has.

VII. CONCLUSION

In probability theory and statistics, the negative binomial distribution is a discrete probability distribution of the number of successes in a sequence of Bernoulli trials before a specified (non-random) number of failures (denoted r) occur. Therefore, we can apply it to real-world problems, outcomes of success and failure may or may not be outcomes, and we ordinarily view as good and bad, respectively.

The negative binomial distribution is one of the probability distributions that can accommodate smooth relationships among other probability distributions. The properties that were outlined can serve as an important bridge to other distributions.

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